Topics in Geometry and Topology

August 18, 2015

Remark. This is the live-texed notes of Topics in Geometry and Topology course held by András Stipsicz in the winter of 2015. Despite all my efforts it still may contain mistakes or typos. The images are mainly copied from the related sections of the book A. Hatcher: Algebraic Topology.

First Lecture, 12th of January

Outline of the course:

1. Simplicial and singular homology
2. Basic homological algebra (chains and homotopies)
3. Relative homology
4. Excision principle (statement only)
5. Degree of maps and CW-homology
6. Cohomology and the corresponding ring structure
7. Universal Coefficient Theorem (statement only)
8. Orientability, Poincaré dual (statement only) and applications
9. Obstruction theory
10. Fiber bundles, vector bundles and principal bundles
11. Classification of vector bundles (statement only)
12. The splitting principle
13. Characteristic classes
14. Curvature and its connection to characteristic classes (if time allows)

Literature:

- Hatcher: Algebraic Topology, Chapter 2 and 3 (for topics up to Orientability)
- Fuchs - Fomenko - Gutenmacher: Homotopic topology (for topics from Obstruction Theory)
- Husemoller: Fiber bundles

Background: It is assumed that the reader is familiar with the basic concepts of general topology such as open-closed sets, compactness, connectedness, etc. Moreover, the notion of fundamental group is assumed and its basic relation to covers.

Convention: In the following, every map is assumed to be continuous except when we explicitly mention that it is not or when it is crucial to prove their continuity.

1 Homology

Aim: To understand manifolds. Maybe it is too much to hope, so instead we would like to construct invariants for topological spaces. These associated objects can be numbers, groups or basically anything that can be "easily" compared on two manifolds. This last assumption mostly rules out the fundamental group with a presentation since it is usually too complicated. However, a finitely generated abelian group – as homologies and cohomologies will be – would be more comfortable for our purpose.

1.1 Simplicial homology

Suppose V is a finite set.

Definition 1.1. A nonempty subset $S \subseteq \mathcal{P}(V)$ is a (combinatorial) *simplicial complex* if

1. $\cup S = V$
2. $A \in S$ and $B \subseteq A$ implies $B \in S$.

The geometric realization of S is the following:

Definition 1.2. An (or "the" if you like it that way) n-dimensional *simplex* Δ is the complex hull of $n+1$ linearly independent vectors in $\mathbb{R}^{n+1}$. In formula: if $e_0, \ldots, e_n \in \mathbb{R}^{n+1}$ then

$$\Delta^n = \left\{ \sum_{i=0}^{n} \lambda_i e_i \mid \lambda_i \in [0,1],\ \sum \lambda_i = 1 \right\}$$

This definition depends on the choice of e_i's of course, but all n-dimensional simplices are homeomorphic. For technical reasons, we fix an n-dimensional simplex in $\mathbb{R}^{n+1}$ denoted by Δ^n for all n: when the e_i vectors are the standard unit-vectors.

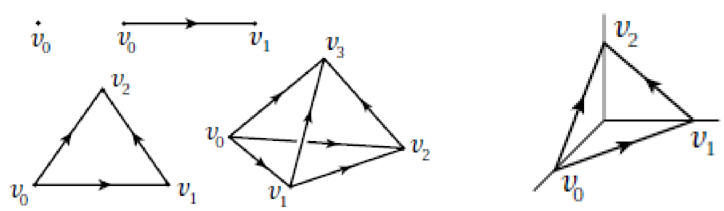


Figure 1.1: Simplices

Definition 1.3. Suppose that S is a simplicial complex ($|V| < \infty$). Its *geometric realization* $Z(S)$ is defined as follows:

- Order the elements of V as $\{v_1, \ldots, v_n\}$. This induces an ordering of every element of S. To include that in our notation, we will write $A = [v_{i_1}, \ldots, v_{i_n}]$.
- Choose basis vectors $e_1, \ldots, e_n$ corresponding to the elements v_i.

Then we define the realization as

$$Z(S) := \bigcup_{A \in S} \text{Convhull}\{e_{i_1}, \ldots, e_{i_k} \mid A = [e_{i_1}, \ldots, e_{i_k}]\}$$

Definition 1.4. A *triangulation* of a topological space X is a homeomorphism of X with some $Z(S)$ for a simplicial complex S.

A more accurate name could be the – weird sounding – simplicialization since we “decomposed” the space into simplices. Anyway, two dimensional simplices are triangles so the name is not too confusing. The plan is to associate homology groups to simplicial complexes hence to triangulated topological spaces. However, there are two problems with this process:

Questions:

1. Does every topological space admit a triangulation?
2. If so, is such a triangulation unique?

The second question should clearly not taken literally: we can always refine a triangulation so it is not unique. The real question is that whether two triangulations give the same invariants, in our case homologies. Still, the answer to both of the questions are no.

Conjecture. *(Hauptvermutung, ~1910) Every pair of triangulations have a common refinement.*

Unfortunately, this conjecture turned out to be false. Even, if we restrict investigation for (topological but not necessarily smooth) manifolds, it is still false. (This is a theorem of 2013!) Anyway, we define the simplicial homology for triangulated spaces and the independency from triangulation will be proved in another way later, by the aid of singular homology.

Definition 1.5. Let S be a simplicial complex. We want to associate abelian groups to S. Recall that we ordered the set $V = \{v_1, \dots, v_N\}$. Now, define

$$C_n(S) := \Big\{ \sum_{i \in I} n_i A_i \mid A_i \in S,\ |A_i| = n+1,\ n_i \in \mathbb{Z},\ I \text{ is finite} \Big\}$$

the free abelian group generated by fixed size elements of S. The elements of $C_n(S)$ are called *chains*. We also define maps between the above spaces:

$$\partial_n : C_{n+1}(S) \to C_n(S)$$

$$\partial_n(A) := \sum_{j=0}^{n} (-1)^j [v_{i_0}, \dots, \hat{v_{i_j}}, \dots, v_{i_{n+1}}]$$

where $A \in S$, $|A| = n+2$ and $A = [v_{i_0}, \dots, v_{i_{n+1}}]$. Now, we can extend the maps linearly. These are called the *boundary maps*.

Lemma 1.6. $\partial_{n-1} \circ \partial_n = 0$

Proof. It is enough to compute the composition on simplices, then linearity implies the statement.

$$\partial_{n-1} \circ \partial_n(A) = \sum_{j<k} \big((-1)^j(-1)^{k-1} + (-1)^k(-1)^j\big)[v_{i_0}, \dots, \hat{v_{i_j}}, \dots, \hat{v_{i_k}}, \dots, v_{n+1}] = 0$$

so the coefficients are indeed zero. □

Definition 1.7. We defined two important subgroups of the chains:

$$Z_n(S) := \{c \in C_n(S) \mid \partial_{n-1}(c) = 0\} = \operatorname{Ker} \partial_{n-1} \le C_n(S)$$

these are called the *cycles*. (The notation Z comes from German.) Similarly, we define

$$B_n(S) = \{c \in C_n(S) \mid \exists d \in C_{n+1}(S) :\ \partial_n(d) = c\} = \operatorname{Im} \partial_n \le C_n(S)$$

called the *boundaries*.

Remark 1.8. Note that these subgroups crucially depend on the triangulation (Easy to see, if one draws a picture).

Definition 1.9. Notice that (by $\partial^2 = 0$) we have that $B_n(S) \leq Z_n(S)$ so it is reasonable to consider the abelian group

$$H_n(S) := Z_n(S)/B_n(S)$$

called the n-th *simplicial homology group* of S.

One should not forget that this object is defined in a purely combinatorial way. The close correspondence with topology will get clear when we prove that it is an invariant of the topological space, no matter how we took a triangulation.

Summary: We associate to S a group $C = \bigoplus C_i(S)$ and an endomorphism $\partial : C \to C$ with the property $\partial^2 = 0$. This (C, ∂) pair is called *chain complex*. In fact, it is a *graded chain complex* because ∂ is graded meaning that it maps C_i into C_{i-1}.

Example 1.10. For homology groups:

1. Let S be a single point $S = \{\text{pt}\}$. Then

$$H_i(\text{pt}) := \begin{cases} 0 & \text{if } i > 0 \\ \mathbb{Z} & \text{if } i = 0 \end{cases}$$

   where – in details – $V = \{v\}$, $S = \{\emptyset, [v]\}$, $C_0 \cong \mathbb{Z}$ and all the other C_i's are zero.

2. Now, let S be a triangle, i.e. $V = [v_0, v_1, v_2]$ consisting of three affine independent vectors, $S = \{\emptyset, [v_0], [v_1], [v_2], [v_0, v_1], [v_0, v_2], [v_1, v_2]\}$. Here

$$C_0 = \langle [v_0], [v_1], [v_2] \rangle \cong \mathbb{Z}^3$$

$$C_1 = \langle [v_0, v_1], [v_0, v_2], [v_1, v_2] \rangle \cong \mathbb{Z}^3$$

   while $\partial[v_i] = 0$ and $\partial[v_{i_0}, v_{i_1}] = (-1)^0[v_{i_0}] + (-1)^1[v_{i_1}] = v_{i_1} - v_{i_0}$ by the definitions. Therefore, $H_1(S) \cong \mathbb{Z}$, $H_0(0) \cong \mathbb{Z}$ and $H_i(S) = 0$ for $i > 1$.

Remark 1.11. We can use different Abelian groups instead of $\mathbb{Z}$ as the coefficients in $C_n(S)$. In particular, we can consider any commutative ring with a unit (e.g. $\mathbb{Z}_2$, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$).

1.2 Singular homology

Let X be a topological space.

Definition 1.12. A continuous map $\sigma : \Delta^n \to X$ is called a *singular simplex*.

The name singular comes from that we only required continuity, so no injectivity, no smoothness hence the image of the maps can look quite "degenerated".

Definition 1.13. We can define the corresponding Abelian group of *chains*:

$$C_n(X) := \left\{ \sum_{i \in I} n_i \sigma_i \mid n_i \in \mathbb{Z},\ \sigma_i \text{ is a singular simplex},\ I \text{ is finite} \right\}$$

If our topological space is too small then this group is horribly huge (not even countably generated) since it contains for example all constant maps and much much more.

Now, analogously to the simplicial case, we want to define the boundary maps. A first guess is

$$\partial_n : C_{n+1}(X) \to C_n(X)$$

$$\partial_n(\sigma) = \sum_{i=0}^{n+1} (-1)^i \sigma|_{[e_0,\dots,\hat{e_i},\dots,e_{n+1}]}$$

but there is a small cheat in the – essentially correct – definition. Namely, the domain of the map $\sigma|_{[e_0,\dots,\hat{e_i},\dots,e_{n+1}]}$ differs on different $\hat{e_i}$'s, they are different subsets of Δ^{n+1} and – at least formally – that is a problem. It is the fundamental problem of "same vs isomorphic" what is not always irrelevant but most of the time it does not matter. So the problem is solved by choosing canonical identifications of $[e_0, \dots, \hat{e_i}, \dots, e_{n+1}]$ with Δ^n. Then the above equation becomes a valid definition.

Definition 1.14. Similarly, the *cycles* and *boundaries* are $Z_n(X) = \operatorname{Ker} \partial_{n-1}$ and $B_n(X) = \operatorname{Im} \partial_n$.

Lemma 1.15. *The map $\partial_{n-1} \circ \partial_n : C_{n+1}(X) \to C_{n-1}(X)$ is zero. (The same proof works as before.)*

Definition 1.16. By the lemma, we can define n-th (singular) homology group of X as $H_n(X; \mathbb{Z}) := Z_n(X; \mathbb{Z})/B_n(X; \mathbb{Z})$. Sometimes $H_n(X)$ is used as a shorthand for $H_n(X; \mathbb{Z})$.

The advantage of this concept is that this construction obviously works for arbitrary topological space and it does not depend on some extra choice so it is an invariant (under homeomorphisms) of X.

Remark 1.17. The intuitive meaning of homologies is that it measures the number of n-dimensional holes on the space X.

Remark 1.18. Notice that

1. We can use other coefficient groups again, as in the simplicial case,
2. We can also use commutative rings,
3. $H_1(X; \mathbb{Z}) \cong \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$ where it is important to note that the left hand side is independent of the base point x_0.

Proposition 1.19. *Let X be a nonempty path-connected topological space. Then $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. By definition $C_{-1}(X; \mathbb{Z}) = 0$ so the 0-th homology group is $H_0(X; \mathbb{Z}) := C_0(X; \mathbb{Z})/\operatorname{Im} \partial_0$. The chains in $C_0(X; \mathbb{Z})$ are nothing more but points in X because $\Delta^0 = \{\text{pt}\}$. Now, consider the should-be-isomorphism

$$\varepsilon : C_0(X; \mathbb{Z}) \to \mathbb{Z} \qquad \sum_i n_i \sigma_i \mapsto \sum n_i \in \mathbb{Z}$$

We prove that $\operatorname{Ker} \varepsilon = \operatorname{Im} \partial_0$ and – by the homomorphism theorem – the statement follows. For the containment $\subseteq$ suppose that $\sum n_i \sigma_i$ satisfies $\sum n_i = 0$. Here, σ_i's are still simply points in X. By $\sum n_i = 0$, we can pair the (signed) points corresponding to σ_i's with multiplicity such that in a pair one point comes with coefficient $+1$ and the other one with -1. Then we can connect these pairs, by the assumption of path-connectedness. The resulting paths have the pair (with signs) as a boundary so the stated containment holds.

The other containment $\operatorname{Ker} \varepsilon \supseteq \operatorname{Im} \partial_0$ can be check on the generators of C_1 where it is completely clear. The proposition follows. ☐

Remark 1.20. The above ε is called the augmentation.

Exercise 1.21. (Homework for 19th of January!)

1. Suppose that $X = \{\text{pt}\}$. Show that $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_i(X; \mathbb{Z}) = 0$ for $i > 0$.
2. If $X = \cup_{i=1}^k X_i$ where the X_i are the path-connected components of X then $H_0(X; \mathbb{Z}) = \mathbb{Z}^k$.

Remark 1.22. Using the sequence

$$\dots \xrightarrow{\partial_n} C_n(X; \mathbb{Z}) \xrightarrow{\partial_{n-1}} C_{n-1}(X; \mathbb{Z}) \xrightarrow{\partial_{n-2}} C_{n-2}(X; \mathbb{Z}) \xrightarrow{\partial_{n-3}} \dots \longrightarrow C_0(X; \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}$$

we get the so called *reduced homology groups* $\tilde{H}_i(X; \mathbb{Z})$. I.e. $\tilde{H}_0$ is defined by $\operatorname{Ker} \varepsilon/\operatorname{Im} \partial_1$ and $\tilde{H}_i = H_i$. Sometimes it is more convenient to use reduced homologies but it is rather just a "linguistic" trick.

1.3 Chain maps

Definition 1.23. Suppose that (C, ∂_C) and (D, ∂_D) are two chain complexes (see Summary under 1.1 for definition). Then a homomorphism $f : C \to D$ is a *chain map* if $\partial_D \circ f = f \circ \partial_C$. Similarly, if $C = \oplus_{i=0}^{\infty} C_i$ and $D = \oplus_{i=0}^{\infty} D_i$ are graded chain complexes then for a map $f = \oplus_{i=0}^{\infty} f_i$ where $f_i : C_i \to D_i$ (i.e. it is a graded map) we expect $\partial_D \circ f = f \circ \partial_C$ to call it a *chain map.*

A chain map induces a homomorphism on homologies, denoted by $f_* : H(C, \partial_C) \to H(D, \partial_D)$. An alternative notation for it is $H(f)$ emphasizing that H is a functor.

Definition 1.24. The chain maps $f, g : (C = \oplus C_i, \partial_C) \to (D = \oplus D_i, \partial_D)$ are *chain homotopic* if there exists a homomorphism $\varphi = \oplus(\varphi_n : C_n \to D_{n+1})$ satisfying $f - g = \partial_D \circ \varphi + \varphi \circ \partial_C$.

Lemma 1.25. *If f and g are chain homotopic then $H(f) = H(g)$.*

Proof. Let $x \in Z_n(C)$ (i.e. $\partial_C x = 0$). Then

$$f(x) = (\partial_D \circ \varphi + \varphi \circ \partial_C)(x) + g(x) = \partial_D(\varphi(x)) + \varphi(\partial_C(x)) + g(x)$$

where $\varphi(\partial_C(x)) = 0$ since $\partial_C x = 0$ and φ is a homomorphism. Therefore f and g are the same up to some elements in $B_n(D)$, or in other words $H(f) = H(g)$. ☐

Second Lecture, 19th of January

Now, let's see how these algebraic notions relate to topology.

Lemma 1.26. *If X and Y are topological spaces and $f : X \to Y$ is a continuous map then it induces $f_\# : C_n(X) \to C_n(Y)$ a chain map.*

Proof. We define $f_\#$ as follows: let $\sigma : \Delta^n \to X$ be a generator of $C_n(X)$ and define $f_\#(\sigma) : \Delta^n \to Y$ by $f_\# = f \circ \sigma$. Obviously, it is a chain map. ☐

Remark 1.27. $f_\#$ should not be confused with f_* which is also an induced map by f but $f_\#$ is a $C(X) \to C(Y)$ map and f_* is a map of homologies $H_*(X) \to H_*(Y)$.

Properties:

- Suppose that $f : X \to Y$ and $g : Y \to Z$. Then $(g \circ f)_* = g_* \circ f_*$.
- $(\mathrm{id}_X)_* = \mathrm{id}_{H(X)}$.

Corollary 1.28. *If $f : X \to Y$ is a homeomorphism then $f_* : H(X) \to H(Y)$ is an isomorphism*

Proof. If f is a homeomorphism then there exists a $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Then by 1.3, we get that $f_* \circ g_* = \mathrm{id}_{H(Y)}$ and $g_* \circ f_* = \mathrm{id}_{H(X)}$ so $f_* : H(Y) \to H(Y)$ is an isomorphism. ☐

Remark 1.29. Recall that if $f, g : X \to Y$ are *homotopic* if there exists $F : X \times [0, 1] \to Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = g$.

This notion allows us to define a generalized notion of equivalence of spaces:

Definition 1.30. A map $f : X \to Y$ is a *homotopy equivalence* if there exists $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

Remark 1.31. It is a strictly weaker notion: there are homotopy equivalent spaces which are not homeomorphic. Indeed, let $X = \{\mathrm{pt}\}$ and $Y = \mathbb{R}^n$. Then let us choose $f : X \to Y$ as $p \mapsto 0$ and $g : Y \to X$ as $v \mapsto p$ for all $v \in Y$. The composition $g \circ f : X \to X$ is already the identity but the other composition $f \circ g : Y \to Y$ is the map $v \mapsto 0$ for all $v \in \mathbb{R}^n$. However, we can define the homotopy $F : Y \times [0, 1] \to Y$ such that $(v.t) \mapsto tv$. Then one can check that this is continuous so X is homotopy equivalent to Y.

Example 1.32. Sometimes homotopy equivalence is not that straightforward to prove, e.g. $\mathrm{Diff}^+(\mathbb{S}^2)$ the group of orientation-preserving diffeomorphisms of the ordinary sphere is homotopy equivalent to $SO(3)$.

Proposition 1.33. *If $f, g : X \to Y$ are homotopic maps then $f_\#, g_\# : C(X) \to C(Y)$ are chain homotopic chain maps.*

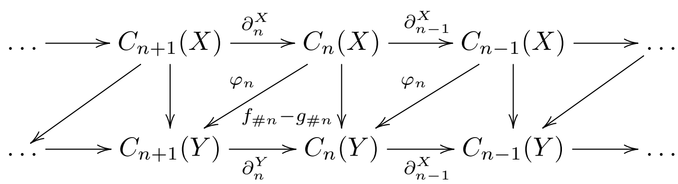
Corollary 1.34. *If X and Y are homotopy equivalent topological spaces then $H_*(X) \cong H_*(Y)$.*

Proof. of the Corollary: By the definitions, we have maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. By the above Proposition 1.33 $(f \circ g)_\#$ and $(\mathrm{id}_Y)_\#$ are chain homotopic so

$$f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{H(Y)}$$

by Lemma 1.25, so we got the statement. □

Proof. of Proposition 1.33: Let us denote the homotopy map by $F : X \times [0,1] \to Y$. The maps $f, g : X \to Y$ gives rise to the chain maps $f_\#, g_\# : C(X) \to C(Y)$. Now, we need maps $\varphi_n : C_n(X) \to C_{n+1}(Y)$ such that $f_\# - g_\# = \varphi \circ \partial_X + \partial_Y \circ \varphi$.
The above formula is a shorthand for $f_{\#n} - g_{\#n} = \varphi_{n-1} \circ \partial_X + \partial_Y \circ \varphi_n$ as $C_n(X) \to C_n(Y)$ maps for all $n \in \mathbb{Z}$. It can be visualized as the commutativity of the following diagram:



Suppose that $\sigma : \Delta^n \to X$ is a singular simplex. Then what should be $\varphi_n(\sigma)$? Our first candidate is the composition

$$\Delta^n \times [0,1] \overset{\sigma \times \mathrm{id}_{[0,1]}}{\longrightarrow} X \times [0,1] \xrightarrow{F} Y$$

However, there is a small problem with it. The domain is not a singular $(n+1)$-simplex. So we will define $\varphi(\sigma)$ as the sum of singular $(n+1)$-simplices in Y using F and the decomposition of $\Delta^n \times [0,1]$ into $(n+1)$-simplices.

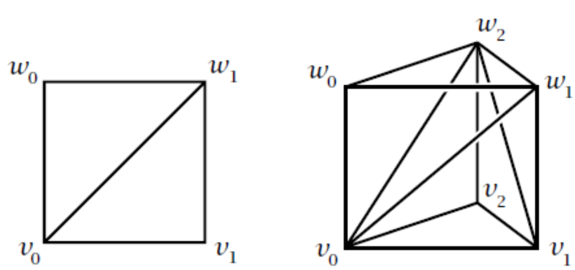


Figure 1.2: Cylinder decomposition

To decompose $\Delta^n \times [0,1]$ we will use the notation $\Delta^n = [0, 1, \dots, n]$ standing for the convex hull of the basis vectors $e_0, e_1, \dots, e_n$. Now, take two copies of Δ^n in $\mathbb{R}^{n+2}$: one copy in the plane $x_{n+2} = 0$ (formed by the vectors $e_0, e_1, \dots, e_n$) and one in the plane $x_{n+2} = 1$ (formed by the vectors $e'_0, e'_1, \dots, e'_n$). We visualize our prism $\Delta^n \times [0,1]$ in this space. So we can see the proof in dimension 2 and 3. For the general argument, we have to check that

$$\Delta^n \times [0,1] = \bigcup_{i=0}^{n} [0, 1, \dots, i, i', (i+1)', \dots, n']$$

where $[0,1,\dots,i,i',(i+1)',\dots,n'] = \mathrm{ConvHull}(e_0,e_1,\dots,e_i,e'_i,e'_{i+1},\dots,e'_n)$.

Homework: (For 26th of January) The above decomposition of $\Delta^n \times [0,1]$ is indeed a decomposition.

Now, we define φ as

$$\varphi(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id}_{[0,1]})\big([0,1,\dots,i,i',(i+1)',\dots,n']\big)$$

We claim that φ provides the chain homotopy between $f_\#$ and $g_\#$. Here, we want to see that $\partial \circ \varphi = g_\# - f_\# - \varphi \circ \partial$ not forgetting that $\varphi(\sigma)$ is a sum of simplices that can have common boundaries as well but – thanks to the introduction of the signs in the above formula – they will cancel out. In details,

$$\partial\varphi(\sigma) = \sum_{i=0}^{n} \Bigg(\sum_{j=0}^{i} (-1)^i(-1)^j F \circ (\sigma \times \mathrm{id}_{[0,1]})[0,1,\dots,\hat{j},\dots,i,i',(i+1)',\dots,n'] +$$

$$+ \sum_{j=i}^{n} (-1)^i(-1)^{j+1} F \circ (\sigma \times \mathrm{id}_{[0,1]})[0,1,\dots,i,i',(i+1)',\dots,\hat{j}',\dots,n'] \Bigg)$$

Observe that the terms with $i=j$ cancel except when $i=j=0$ or $i=j=n$ and these two terms give

$$F \circ (\sigma \times \mathrm{id}_{[0,1]})[0',1',\dots,n'] = g \circ \sigma = g_\#(\sigma)$$

$$F \circ (\sigma \times \mathrm{id}_{[0,1]})[0,1,\dots,n] = f \circ \sigma = f_\#(\sigma)$$

Besides the terms with $i \neq j$ give the terms in $\varphi \circ \partial$ since

$$\varphi \circ \partial(\sigma) = \sum_{i=0}^{n} \Bigg(\sum_{j=0}^{i-1} (-1)^i(-j)^j F \circ (\sigma \times \mathrm{id}_{[0,1]})[0,1,\dots,i,i',(i+1)',\dots,n'] +$$

$$+ \sum_{j=i+1}^{n} (-1)^{i-1}(-j)^j F \circ (\sigma \times \mathrm{id}_{[0,1]})[0,1,\dots,\hat{j},\dots,i,i',(i+1)',\dots,n'] \Bigg)$$

so the calculations complete the proof. □

1.4 Mayer-Vietoris sequence

Theorem 1.35. (Mayer-Vietoris, ~1930) *Suppose that $X = A \cup B = \mathrm{int}A \cup \mathrm{int}B$. Then there exists a long exact sequence (in short, LES)*

$$\dots \xrightarrow{\delta_n} H_n(A \cap B) \xrightarrow{f_n} H_n(A) \oplus H_n(B) \xrightarrow{g_n} H_n(X) \xrightarrow{\delta_{n-1}} H_{n-1}(A \cap B) \longrightarrow \dots$$

where exactness means that at every term of the sequence, the image of the arriving map is exactly the kernel of the starting map.

Remark 1.36. The order can be easily memorized because the map f_n is basically induced by the maps $\sigma_A : A \cap B \hookrightarrow A$, $\sigma_B : A \cap B \hookrightarrow B$, in fact $f_n = \big((\sigma_A)_*, -(\sigma_B)_*\big)$ and similarly, the map g_n is produced by $A \overset{i_1}{\hookrightarrow} X$ and $B \overset{i_2}{\hookrightarrow} X$ by $g_n = (i_1)_* + (i_2)_*$.

Definition 1.37. Suppose that L, M, N are Abelian groups together with the following maps:

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

Then the sequence above is called a *short exact sequence* (in short, SES) if it is exact in every term, i.e. if α is a monomorphism, β is an epimorphism and $\mathrm{Ker}\,\beta = \mathrm{Im}\,\alpha$.

Proposition 1.38. *Suppose that $L = \oplus L_n$, $M = \oplus M_n$ and $N = \oplus N_n$ graded chain complexes (with boundary maps, only mentioned implicitly in the name chain complex) and*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

is a short exact sequence. Then this short exact sequence gives rise to long exact sequence

$$\dots \xrightarrow{\delta_n} H_n(L) \xrightarrow{H_n(\alpha)} H_n(M) \xrightarrow{H_n(\beta)} H_n(N) \xrightarrow{\delta_{n-1}} H_{n-1}(L) \longrightarrow \dots$$

Proof. The proof involves many steps, most of which are part of the homework.

First, let's define δ_{n-1}. Take an element $x \in N_n$ with $\partial^N x = 0$, it gives rise to $[x] \in H_n(N)$. We know that β is surjective so there exists a $y \in M_n$ such that $\beta(y) = x$. Let $z = \partial^M y \in M_{n-1}$. Notice that β is a chain map and $\partial^N x = 0$ so $\beta(z) = \beta(\partial^M y) = \partial^N \beta(y) = \partial^N x = 0$. Therefore, by $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, we have an element $u \in L_{n-1}$ such that $\alpha(u) = z$. In diagram,

$$\begin{array}{ccccc} & & y \in M_n & \xrightarrow{\beta} & N_n \ni x \\ & & \downarrow{\scriptstyle \partial^M} & & \downarrow{\scriptstyle \partial^N} \\ u \in L_{n-1} & \xrightarrow{\alpha} & z \in M_{n-1} & \xrightarrow{\beta} & N_{n-1} \ni 0 \end{array}$$

Now, what is $\partial^L u$? Since α is injective, it is enough to determine the following:

$$\alpha(\partial^L u) = \partial^M \alpha(u) = \partial^M z = \partial^M(\partial^M y)$$

what is zero on the right because $\left(\partial^M\right)^2 = 0$. So we got that $\partial^L u = 0$ hence u is a cycle.

Note that in the definition we made two choices: first, we picked $x \in N_n$ representing the homology class $[x]$, and second we picked $y \in M_n$ neither of which is (necessarily) unique. So we should show that $[u] \in H_{n-1}(L)$ is independent of the choice of x from the homology class $[x]$ and of the choice of y in the above process.

Homework: (for 26th of January)

1. Show that $[u]$ is independent of the choice of y.
2. Show that δ_{n-1} is a well-defined map, i.e. independent of the choice of the cycle x.
3. Show that δ_{n-1} is a homomorphism.

Now, we prove that the long sequence is exact:

On one hand, we have to check that $\operatorname{Im} H_n(\alpha) = \operatorname{Ker} H_n(\beta)$. The containment $\subseteq$ is easy since $\operatorname{Im}\alpha = \operatorname{Ker}\beta$ in particular, $\beta \circ \alpha = 0$ so $H_n(\beta) \circ H_n(\alpha) = H_n(\beta \circ \alpha) = 0$. For the $\supseteq$ part: take $[m] \in \operatorname{Ker} H_n(\beta)$. It means that m is a cycle so that $\beta(m) = \partial^N n$ for some $n \in N_{n+1}$. We know that β is surjective so there exists an $m' \in M_n$ such that $\beta(m') = n$. This shows that $\beta(m - \partial^M m') = 0$ because

$$\beta(m) = \partial^N n = \partial^N \beta(m') = \beta(\partial^M m')$$

So $m - \partial^M m' \in \operatorname{Ker}(\beta) = \operatorname{Im}\alpha$ i.e. there exists an $l \in L$ such that $\alpha(l) = m - \partial^M m'$. This l is a cycle because

$$\alpha(\partial^L l) = \partial^M \alpha(l) = \partial^M(m - \partial^M m') = 0$$

since m was a cycle, but α is injective so $\partial^L l = 0$. Now, what we need is that the class of l maps to $[m]$ by $H_n(\alpha)$ but it is clear since

$$H_n(\alpha)\big([l]\big) = \big[\alpha(l)\big] = \big[m - \partial^M m'\big] = [m]$$

so we got the exactness at $H_n(M)$.

We also have to check $\operatorname{Im} H_n(\beta) = \operatorname{Ker} \delta_{n-1}$. First, we prove $\subseteq$: Take an element $[x] \in \operatorname{Im} H_n(\beta)$ then we can take a cycle $m \in M$ such that $\beta(m) = x \in N$ (maybe after choosing another representing x). Since β is a chain map, x is a cycle, we can pick $y = m$ in the definition of δ_{n-1}. However, m is a cycle so $\partial^M m = 0$ so following the definition of $\delta_{n-1} = "\beta^{-1}" \circ \partial^M \circ "\alpha^{-1}"$ we get that $\delta_{n-1}([m]) = 0$.

To verify the other containment $\supseteq$ we only need to check that $H_{n-1}(\alpha) \circ \delta_{n-1} = 0$ and it easily follows from the definition.

Homework: (For 26th of January) Choose and prove one of the two missing parts of the proof of exactness, i.e. $\operatorname{Ker} \delta_{n-1} \subseteq \operatorname{Im} H_n(\beta)$ and $\operatorname{Ker} H_n(\alpha) \subseteq \operatorname{Im} \delta_{n-1}$.

Provided that we accept these short exercises, we are done with the proof of the Proposition. □

Now, let's get back to the proof of the Mayer-Vietoris sequence.

Definition 1.39. For the two chain complexes $C_n(A)$ and $C_n(B)$ we define $C_n(A) + C_n(B)$ the set of all singular chains in X which are linear combinations of singular chains in A or B, i.e.

$$C_n(A) + C_n(B) := \{z \in C_n(X) \mid z = \sum n_i \sigma_i \text{ when } \sigma_i \text{ maps either to } A \subseteq X \text{ or to } B \subseteq X\}$$

Proof. of Mayer-Vietoris, Theorem 1.35: Consider the short sequence

$$0 \longrightarrow C_n(A \cap B) \longrightarrow C_n(A) \oplus C_n(B) \longrightarrow C_n(A) + C_n(B) \longrightarrow 0$$

where we map $\sigma \in C_n(A \cap B)$ into $(\sigma, -\sigma)$ and $(\sigma, \tau) \in C_n(A) \oplus C_n(B)$ into $\sigma + \tau \in C_n(A) \oplus C_n(B)$. This is, in fact, a short exact sequence. By Proposition 1.38, we get a long exact sequence

$$\dots \xrightarrow{\delta_n} H_n(A \cap B) \xrightarrow{f_n} H_n(A) \oplus H_n(B) \xrightarrow{\tilde{g}_n} H_n(C(A) + C(B)) \xrightarrow{\delta_{n-1}} H_{n-1}(A \cap B) \longrightarrow \dots$$

where f_n is the mentioned one in the statement and $\tilde{g}_n$ is more or less the map we require there but at the third place we have $H_n(C_n(A) + C_n(B))$ and not $H_n(X)$. Fortunately, these two groups are naturally isomorphic, this will come next time.

THIRD LECTURE, 22TH OF JANUARY

Claim 1.40. $H_n(X) \cong H_n\big(C(A) + C(B)\big)$

Proof. The idea is to subdivide simplices $\sigma : \Delta^n \to X$ so that all the parts fall in either A or B. The way is to apply barycentric subdivision what we define inductively. For $n = 0$: we have no choice, it is $\Delta^0 \to \Delta^0$. For $n = 1$: Δ^1 divides into two 1-simplices, cutting the original one into halves.

Generally, assume that we have a barycentric subdivision of simplices of dimension $n - 1$ and consider Δ^n. The subdivision is defined as: take p the center of mass ($p = \frac{1}{n+1} \sum_i v_i$) and write Δ^n as the union of simplices of the form $[p, w_0, \dots, w_{n-1}]$ where $[w_0, w_1, \dots, w_n]$ is a simplex (with notation defined in the proof of Proposition 1.33) in the barycentric subdivision of $\partial \Delta^n$.

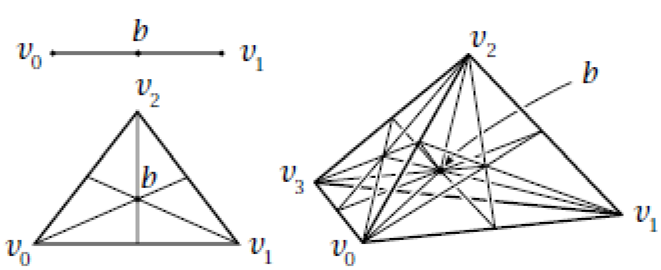


Figure 1.3: Barycentric subdivision

Here, we claim (but do not prove) that the diameter of the subdivision of a simplex in the barycentric subdivision is $\frac{n}{n+1}$-times the diameter of the original n-simplex. So the diameter can drop below any ε by repeated application of the above division algorithm.

By the Vietoris-Lebesgue lemma, we know that for an open cover of a compact metric space there exists an ε such that all the ε-neighborhoods are contained in some element of the cover. We can apply it on $\mathrm{int}A$ and $\mathrm{int}B$ so in a fine enough subdivision of the simplex, every tiny simplex falls into either $\mathrm{int}A$ or $\mathrm{int}B$, hence every n-simplex got divided into sum of elements of $C_n(A)$ and $C_n(B)$. This means $C_n(X) \cong C_n(A)+C_n(B)$ for all n.

Now, we only have to show that these isomorphisms together give some kind of a chain isomorphism so the identification "survives" taking homologies. In fact, we need that the above defined $C(X) \to C(A)+C(B) \subseteq C(X)$ is homologous to $\mathrm{id}_{C(X)}$ what is fortunately true. Indeed, if we take a singular simplex $\Delta^n \to X$ and its subdivision $D(\Delta^n) \to X$ then one can construct a map $F : \Delta^{n+1} \to X$ such that its boundary is the difference of the previous two n-chains. □

The claim proves the statement. □

Proposition 1.41. *For $n > 0$, $H_i(\mathbb{S}^n;\mathbb{Z}) = 0$ if $i \neq 0$ or n. In the exceptional cases, we have $H_n(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z}$ and $H_0(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z}$.*

Proof. We want to use the Mayer-Vietoris sequence, so let's find A and B. The first guess is the upper hemisphere $A = \{x_n \geq 0\}$ and the lower hemisphere $B = \{x_n \leq 0\}$ where $\mathbb{S}^n = \{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1\}$. However, in this case $\mathrm{int}A \cup \mathrm{int}B \neq \mathbb{S}^n$ so we have to thicken the equator by ε. This solves our problem and now Mayer-Vietoris is applicable.

Now, we compute the homologies in the long exact sequence. For this purpose, note that A and B are homeomorphic to the disk D^n which is homotopy equivalent to the point so we know their homologies. Besides, $A \cap B$ is homeomorphic to $\mathbb{S}^{n-1} \times (-\varepsilon,\varepsilon)$ what is homotopy equivalent to $\mathbb{S}^{n-1}$ so we know its homologies by the induction hypothesis. Therefore, we got the following long exact sequence:

$$\dots \longrightarrow H_i(D^n) \oplus H_i(D^n) \longrightarrow H_i(\mathbb{S}^n) \xrightarrow{\delta_{i-1}} H_{i-1}(\mathbb{S}^{n-1}) \longrightarrow H_{i-1}(D^n) \oplus H_{i-1}(D^n) \longrightarrow \dots$$

If $i-1 > 0$ the appearing zeros give that $H_i(\mathbb{S}^n) \overset{\cong}{\to} H_{i-1}(\mathbb{S}^{n-1})$ and that makes the induction work. We only have to start the induction.

Recall that $\mathbb{S}^0 = \{\mathrm{pt}\} \overset{*}{\cup} \{\mathrm{pt}\}$ so $H_0(\mathbb{S}^0) \cong \mathbb{Z} \oplus \mathbb{Z}$. However, we can consider $\tilde{H}_i(X;\mathbb{Z})$ instead of $H_i(X;\mathbb{Z})$ (see Remark 1.22) and then the above argument works for $i = 0$ as well. So we are done. □

Corollary 1.42. *$\mathbb{S}^n$ and $\mathbb{S}^m$ are homotopy equivalent if and only if $n = m$. (It is also true with homeomorphism.)*

Corollary 1.43. *$\mathbb{R}^n \times \mathbb{R}^m$ are homeomorphic if and only if $n = m$. (It is not true with homotopy equivalence: $\mathbb{R}^n$'s are homotopy equivalent for all n.)*

Proof. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism and pick a point $\mathrm{pt} \in \mathbb{R}^n$. The define $f|_{\mathbb{R}^n \setminus \{\mathrm{pt}\}} : \mathbb{R}^n \setminus \{\mathrm{pt}\} \to \mathbb{R}^m \setminus \{f(\mathrm{pt})\}$. It can be easily seen that these spaces are homotopy equivalent to $\mathbb{S}^{n-1}$ and $\mathbb{S}^{m-1}$. Therefore, we got that if $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ then $\mathbb{S}^{n-1}$ is homotopy equivalent to $\mathbb{S}^{m-1}$. By the previous corollary, it implies $n = m$. □

Remark 1.44. There is a theorem that states that if X is a differentiable manifold that is homeomorphic to $\mathbb{R}^n$ and $n \neq 4$ then X is diffeomorphic to $\mathbb{R}^n$. Moreover, in $n = 4$ it is even false. The core reason for this is the non-simpleness of $SO(4)$ which is the only $SO(n)$ that is not simple.

1.5 Relative homology

Definition 1.45. Suppose that $A \subseteq X$ is a pair of topological spaces. Consider $C_n(X, A) := C_n(X)/C_n(A)$ with the boundary maps $\partial_{(X,A)} : C_n(X, A) \to C_{n-1}(X, A)$ by $\alpha \mapsto \partial_X(\alpha)$ where $\alpha \in C_n(X)$ represents an element of $C_n(X, A)$. And yes, we have to worry about that it is well defined but it obviously is.

This gives a chain complex $(C_n(X, A), \partial_{(X,A)})$ with homology $H_n(X, A; \mathbb{Z})$. It also fits into a long exact sequence coming from a pretty natural short exact sequence:

$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \longrightarrow C_n(X)/C_n(A) \longrightarrow 0$$

so it gives

$$\dots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\delta_{n-1}} H_{n-1}(A) \longrightarrow \dots$$

Question. *What is δ_{n-1} in this case?*

Let's follow the definition: we took an element $a \in C_n(X, A)$ representing the class $[a] \in H_n(X, A)$. Then we chose a preimage $b \in C_n(X)$ i.e. it looks like a plus an element of $C_n(A)$. Then we had to form its boundary $\partial_X b$ that is "visually clearly" in $C_{n-1}(A)$ since a was a cycle in $C_n(X, A)$ so it can have boundary only in A (and something in $C_{n-1}(A)$ does not change that). So taking its preimage in $C_{n-1}(A)$ is just some formal manipulation, it is still the boundary $\partial_X b$ just considered in $C_{n-1}(A)$. So it is an easy way of visualizing the connecting homomorphism in this special case.

Theorem 1.46. (Excision principle) *Suppose that $Z \subseteq A \subseteq X$ and $\bar{Z} \subseteq \text{int}A$. Then the map $(X \backslash Z, A \backslash Z) \overset{\iota}{\hookrightarrow} (X, A)$ induces an isomorphism $i_* : H_n(X \backslash Z, A \backslash Z; \mathbb{Z}) \overset{\cong}{\to} H_n(X, A; \mathbb{Z})$ on homology. Geometrically, erasing Z from both A and X does not change the relative homology.*

Question. *What properties characterize the functor $(X, A) \to H_*(X, A; \mathbb{Z})$?*

Answer: (Eilenberg-Steenrod axioms) The required properties are: additivity, Excision principle, Mayer-Vietoris, induced maps on homotopic maps are the same, and finally the homology of the point should be the usual one. Excluding this last axiom allows many other types of homology theories with similar properties.

Theorem 1.47. *Let $n, k, i \in \mathbb{N}$ be arbitrary provided that $k \leq n - 1$.*

1. *Suppose $h : D^k \to \mathbb{S}^n$ is a topological embedding (i.e. h is a homeomorphism $h(D^k)$. This implies that $\tilde{H}_i(\mathbb{S}^n \backslash h(D^k); \mathbb{Z}) = 0$*
2. *If $h : \mathbb{S}^k \to \mathbb{S}^n$ is a topological embedding then $\tilde{H}_i(\mathbb{S}^n \backslash h(\mathbb{S}^k); \mathbb{Z}) = \tilde{H}_i(\mathbb{S}^{n-k-1}; \mathbb{Z})$.*

Corollary 1.48. *$n = 2$, $k = 1$ then Corollary 2) gives that $\tilde{H}_0(\mathbb{S}^2 \backslash h(\mathbb{S}^1)) = \tilde{H}_0(\mathbb{S}^0; \mathbb{Z}) \cong \mathbb{Z}$ so $H_0(\mathbb{S}^2 \backslash h(\mathbb{S}^1)) = \mathbb{Z}^2$ which is the Jordan curve theorem: $\mathbb{S}^2 \backslash h(\mathbb{S}^1)$ has two compact.*

Proof. (of Theorem 1.47) We proceed by induction on k: the case $k = 0$ is clear. Now, consider $D^k \cong I^k = [0, 1]^k$ and define $A_+ = \mathbb{S}^n \backslash h(I^{k-1} \times [0, \frac{1}{2}])$ and $A_+ = \mathbb{S}^n \backslash h(I^{k-1} \times [\frac{1}{2}, 1])$. One can check that $A_+ \cap A_- = \mathbb{S}^n \backslash h(I^k)$ and $A_+ \cup A_- = \mathbb{S}^n \backslash h(I^{k-1})$ (h being homeomorphism is used here). So, by induction, we know that the homologies of $A_+ \cup A_-$ are zero and we are heading for $A_+ \cap A_-$. (Fortunately, the intersection is open so we can apply it.) Mayer-Vietoris gives: $\tilde{H}_i(\mathbb{S}^n \backslash h(I^k)) \cong \tilde{H}_i(A_+) \oplus \tilde{H}_i(A_-)$ so we now only have to see that these are already zero.

Suppose that $\alpha \in \tilde{H}_i(\mathbb{S}^n \backslash h(I^k))$ is not a boundary hence it is not a boundary in either A_+ or A_-. Now, use the same principle for further subdivisions of the interval $[0, 1]$ as in the proof of the Mayer-Vietoris Theorem (1.35). So by iteration we can get a sequence of intervals $I_1 \supseteq I_2 \supseteq \dots \supseteq I_j \supseteq \dots$ such that $\cap I_j = \{p\}$ and α is not a boundary in $\mathbb{S}^n \backslash h(I^{k-1} \times I_j)$. Intuitively, we added the missing subsets of $\mathbb{S}^n$ in several steps and we realized that it cannot be a boundary in any step. On the other hand, in $\mathbb{S}^n \backslash h(I^{k-1} \times \{p\})$, α is a boundary by induction so it is a contradiction: if α is a boundary here then – by compactness implying positive distance – it should be a boundary in some finite step.

In the second statement, we – again – use induction on k. If $k = 0$ then $\tilde{H}_i(\mathbb{S}^n \backslash \mathbb{S}^0) \cong \tilde{H}_i(\mathbb{R}^n \backslash \{\text{pt}\}) \cong \tilde{H}_i(\mathbb{S}^{n-1})$ by homotopy equivalence. Let us decompose $\mathbb{S}^k = D^k_+ \cup D^k_-$ as the ε-neighborhood of the upper and lower hemisphere. We denote the decomposition of the subsets as $B_+ := \mathbb{S}^n \backslash h(D^k_+)$ and $B_- := \mathbb{S}^n \backslash h(D^k_-)$. By part 1) we know that $\tilde{H}_i(B_\pm) = 0$. By the Mayer-Vietoris Theorem 1.35 and $B_+ \cap B_- \sim \mathbb{S}^n \backslash h(\mathbb{S}^{n-1})$ (homotopy equivalence) we get

$$\tilde{H}_i\big(\mathbb{S}^n \backslash h(\mathbb{S}^k)\big) \cong \tilde{H}_{i+1}\big(\mathbb{S}^n \backslash h(\mathbb{S}^{k-1})\big) \overset{\text{induction}}{=} \tilde{H}_{i+1}(\mathbb{S}^{n-k+1-1}) \cong \tilde{H}_{i+1}(\mathbb{S}^{n-k}) \cong \tilde{H}_i(\mathbb{S}^{n-k-1})$$

The statement follows. □

Definition 1.49. Suppose that $f : \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map. Then f induces a map $f_* : H_n(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z} \to \mathbb{Z} \cong H_n(\mathbb{S}^n; \mathbb{Z})$. Every such map is a multiplication by the integer $f_*(1) \in \mathbb{Z}$. This is called the *degree* $\deg f$ of f. This integer corresponding to a homology class is defined only up to sign but if we use the same isomorphism $H_n(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z}$ in both places in the definition then the degree is already well defined and independent of the choice of the isomorphism.

Properties:

1. If $f = \text{id}$ then $\deg f = 1$.
2. If f is not surjective then $\deg f = 0$. (i.e. f factors through $\mathbb{S}^n \to D^n \hookrightarrow \mathbb{S}^n$ so f_* is zero since D^n has zero homologies)
3. If f has no fixed point then $\deg f = (-1)^{n+1}$.

Proof. Sketch on part 3): We assumed that $f(x) \neq x$ for all $x \in \mathbb{S}^n$. Then take the normalization of the map $(t, x) \mapsto (1-t)f(x) - t \cdot x$ so it takes values in $\mathbb{S}^n$. To make it work, we have to check that it does not gives zero on any point of $\mathbb{S}^n$.

Homework:

1. Check that the above defined f_t is well defined, i.e. $||(1-t)f(x) - t \cdot x|| \neq 0$ for all $t \in [0, 1]$, $x \in \mathbb{S}^n$.
2. Verify that the degree of a reflection to a plane is (-1).

By these, the statement follows since the antipodal map is the composition of $n + 1$ reflections and f_1 is the antipodal map. Here we used that the degree of composed maps is the product of the degrees but it is trivial by $(f \circ g)_* = f_* \circ g_*$ (see 1.3). □

1.6 CW-complexes

Definition 1.50. The topological space X is a *CW-complex* if it can be constructed inductively as follows: X^0 is a discrete space (called the 0-cells). In the inductive step, if X^i is already constructed for all $i < n$ then the *n-skeleton* X^n of X is defined as

$$X^n = (X^{n-1} \overset{*}{\cup} \amalg D_i) / \{x \sim \varphi_\alpha(x) \mid x \in \partial D^n_\alpha\}$$

where $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ are the called the *gluing maps*.

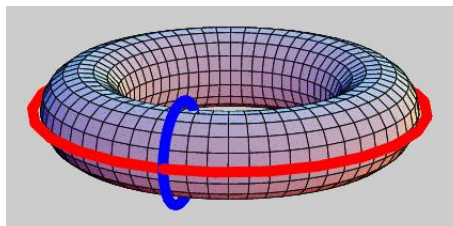
So the base set of X is $\cup X^n$ where we require that each cell D^n_α has the property that $\overline{D^n_\alpha}$ intersects only finitely many cells (it is called the closure finite property, that's the 'C' in the name). The topology on X is the weak topology (hence the 'W' in the name) given by the embeddings $X^n \hookrightarrow X$ i.e. $A \subseteq X$ is open if and only if $A \cap X^n$ is open for all n.

Fourth Lecture, 26th of January

Example 1.51. For CW-complexes:

1. $\mathbb{S}^n$ is a CW-complex: X^0 is a point, as all the higher X^i's up to $i \leq n-1$. In the n-th step, we attach a D^n where the boundary is mapped into the point. One can easily see that the result is homeomorphic to $\mathbb{S}^n$.

2. The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is also a CW-complex: $X^0 = \{\mathrm{pt}\}$, $X^1 = \mathbb{S}^1 \vee \mathbb{S}^1$ where $\vee$ means the union where they meet at exactly the one point pt. This is given by two D^1's where the boundaries are mapped into pt. X^2 will be the whole space by attaching one D^2 with the boundary map.



Remark 1.52. Similar CW-complex decompositions exists for all closed surfaces, where 'similar' means that it contains one point, several edges and one 2-cell.

Remark 1.53. If a topological space is not homeomorphic to a CW-complex then it "must be very sick" from a geometrical point of view. The reason is that the definition of topological space is a bit two wild. It is necessary since, for example, in subjects as algebraic geometry or topological vector spaces such examples may appear. Now, they will not be important in this course. So – at least intuitively – we always restrict ourselves to CW-complexes.

Theorem 1.54. *(without proof, not necessarily hard statements) Suppose that X is a CW-complex. Then*

1. $H_i(X^n; X^{n-1}; \mathbb{Z}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z}^{|\{n\text{-cells}\}|} & \text{if } k = n \end{cases}$

2. $H_k(X^n) = 0$ *for all* $k > n$.

3. *The embedding* $i : X^n \to X$ *induces an isomorphism* $i_* : H_k(X^n) \to H_k(X)$ *once* $k < n$.

1.7 CW-homology

Definition 1.55. Let $C_n^{CW}(X) := H_n(X^n; X^{n-1}; \mathbb{Z})$ i.e. the free abelian group given in Theorem 1.54 with the boundary map $\mathrm{d}_{n-1} : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ defined by the composition of the morphisms in the diagram

$$H_n(X^n; X^{n-1}; \mathbb{Z}) \xrightarrow{\delta_{n-1}} H_{n-1}(X^{n-1}; \mathbb{Z}) \xrightarrow{\gamma_{n-1}} H_{n-1}(X^{n-1}; X^{n-1}; \mathbb{Z})$$

where γ_{n-1} is induced by the embedding and δ_{n-1} is the connecting morphism given by the long exact sequence of the homologies of the pair (X^n, X^{n-1}).

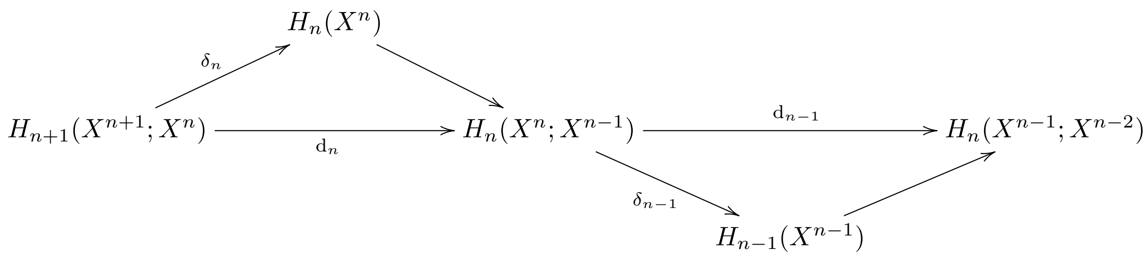
Remark 1.56. A simplicial complex naturally gives a unique CW-complex (of quite special form since here a boundary simplex is (linearly) mapped to exactly one other simplex).

Proposition 1.57. $(C_*^{CW}(X), \mathrm{d}_*)$ *is a chain complex, that is* $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$

Moreover, $H_*(C_*^{CW}, \mathrm{d}_*) \cong H_*(X; \mathbb{Z})$ *where the second* H_* *stands for the singular homology.*

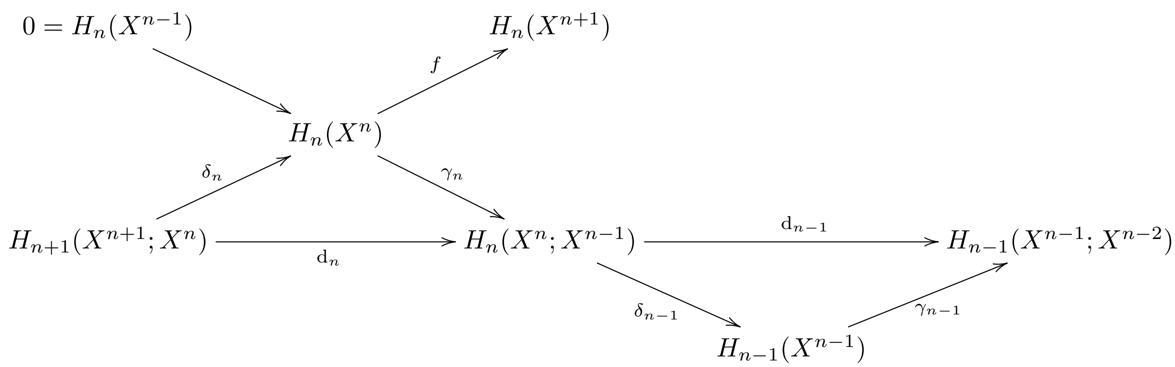
Remark 1.58. The 'slogan' of this proposition is that $C_n^{CW}(X)$ may depend on the CW-decomposition of the space but the homologies already does not. Besides, the resulting homology is exactly the singular homology restricted to CW-complexes. Therefore, we drop CW and $\mathbb{Z}$ from the notation, for the sake of simplicity. The gain of this notion is that it is computable, unlike singular homology.

Proof. We can expand the definitions of $\mathrm{d}_{n-1} \circ \mathrm{d}_n$ in the form of the following diagram:



Here, one can realize that $\mathrm{d}_{n-1} \circ \mathrm{d}_n = (\gamma_{n-1} \circ \delta_{n-1}) \circ (\gamma_n \circ \delta_n) = \gamma_{n-1} \circ (\delta_{n-1} \circ \gamma_n) \circ \delta_n = 0$ by the exactness of the Mayer-Vietoris long exact sequence. So $(C_*^{CW}(X), \mathrm{d}_*)$ is indeed a chain complex.

To get the exactness, extend the previous diagram a bit along the used long exact sequences:



Here, one can observe that the continuation of the long exact sequence with f is exactly $H_n(X^{n+1}, X^n)$ which is zero by Theorem 1.54. Therefore, f is surjective on $H_n(X^{n+1})$. Moreover, $H_n(X^{n+1}) \cong H_n(X)$ by the third part of the same statement. So we found the singular homology, and what we want to compute is that it is $H_n(C^{CW}(X)) := \operatorname{Ker} \mathrm{d}_{n-1} / \operatorname{Im} \mathrm{d}_n$. For this, observe the following:

- γ_n is injective (since $H_n(X^{n-1})$ is zero and contains the kernel of the γ's by exactness) so $\operatorname{Im} \delta_n$ maps isomorphicly to $\operatorname{Im}(\gamma_n \circ \delta_n) = \operatorname{Im}(\mathrm{d}_n)$.
- γ_{n-1} is injective (because, similarly, $H_{n-2}(X^{n-2})$ is zero) hence $\operatorname{Ker} \delta_{n-1} = \operatorname{Ker} \mathrm{d}_{n-1}$.
- While $H_n(X) = H_n(X^n) / \operatorname{Im} \delta_n$ since f is surjective.

Collecting these facts together we get

$$\operatorname{Ker} \mathrm{d}_{n-1} = \operatorname{Ker} \delta_{n-1} = \operatorname{Im} \gamma_n \cong H_n(X^n) \qquad \text{and} \qquad \operatorname{Im} \mathrm{d}_n \cong_{\gamma_n} \operatorname{Im} \delta_n \subseteq H_n(X^n)$$

therefore $\operatorname{Ker} \mathrm{d}_{n-1} / \operatorname{Im} \mathrm{d}_n \cong \operatorname{Coker} \delta_n = \operatorname{Im} f \cong H_n(X)$ and we got the required isomorphism. ☐

Question: Are we done with the project making singular homology computable? We are not: the boundary maps are still defined by singular homology and we should exclude all such "non-computable" parts. So the next goal is to find a combinatorial description of the boundary maps between $C_n^{CW}(X)$'s.

Theorem 1.59. (Cellular boundary formula) *Let X be a CW-complex where the n-th skeleton X^n is decomposed as $X^n = X^{n-1} \cup_\varphi \bigcup_{\alpha \in A} e^n_\alpha$ and $X^{n-1} = X^{n-2} \cup_\varphi \bigcup_{\beta \in B} e^{n-1}_\beta$. (This is just a notation for the n-cells, not an extra information.) Then the boundary of a cell is*

$$\mathrm{d}_{n-1}(e^n_\alpha) = \sum_{\beta \in B} d_{\alpha\beta} e^{n-1}_\beta$$

where the coefficients are

$$d_{\alpha\beta} := \deg\Big(\mathbb{S}^{n-1} \cong \partial e^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\psi_\beta} e^{n-1}_\beta / \partial e^{n-1}_\beta \cong \mathbb{S}^{n-1}\Big)$$

where φ_α is the gluing map and ψ_β is the quotient map given by collapsing the whole $X^{n-1} \backslash e^{n-1}_\beta$ to a point.

Remark 1.60. It still uses singular homology since it is there in the definition of the degree. However, there we only used singular homology of spheres so it is still a lot easier. Moreover, degree have another characterization in terms of differential topology so we can completely exclude singular homology from the computations.

Proof. To determine the coefficient $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ let's collect the maps that are used in its definition:

$$\begin{CD}
H_{n-1}(\partial e^n_\alpha) \cong H_{n-1}(\mathbb{S}^{n-1}) @>{\Delta_{\alpha\beta *}}>> H_{n-1}(\mathbb{S}^{n-1}) \cong H_{n-1}(\partial e^{n-1}_\beta) \\
@V{\varphi_{\alpha *}}VV @AA{\psi_{\beta *}}A \\
H_{n-1}(X^{n-1}) @>{\psi}>> H_{n-1}(X^{n-1}/X^{n-2})
\end{CD}$$

where ψ is the map induced by the collapsing of X^{n-1} onto X^{n-1}/X^{n-2}. The long exact sequences for the pairs $(e^n_\alpha, \partial e^n_\alpha)$ (X^{n-1}, X^{n-2}) and $(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})$ give isomorphisms since every third term of the sequence is zero (by triviality or using Theorem 1.54). Moreover, we can use the long exact sequence of the pair (X^n, X^{n-1}). Hence, we can replace the appearing groups in the following way (using reduced homologies to include the case $n = 1$):

$$\begin{array}{ccccc}
H_n(e^n_\alpha, \partial e^n_\alpha) & \xrightarrow{\cong} & \tilde{H}_{n-1}(\partial e^n_\alpha) & \xrightarrow{\Delta_{\alpha\beta *}} & \tilde{H}_{n-1}(\partial e^{n-1}_\beta) \\
\Big\downarrow{\scriptstyle \Phi_{\alpha *}} & & \Big\downarrow{\scriptstyle \varphi_{\alpha *}} & & \Big\uparrow{\scriptstyle \psi_{\beta *}} \\
H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & \tilde{H}_{n-1}(X^{n-1}) & \xrightarrow{\psi} & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \\
& \searrow{\scriptstyle \mathrm{d}_{n-1}} & \Big\downarrow{\scriptstyle \cong \gamma_{n-1}} & & \Big\downarrow{\scriptstyle \cong} \\
& & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\cong} & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{array}$$

where Φ_α is the characteristic map of the cell e_α, i.e. the image of $[e^n_\alpha] \in H_n(e^n_\alpha; \partial e^n_\alpha)$ corresponding to the cell e^n_α under $\Phi_{\alpha *}$ is $[e^n_\alpha] \in H_n(X^n; X^{n-1})$. One can notice that the composition $\partial_n \circ \gamma_{n-1}$ is d^{CW}_{n-1} by definition and that $H_{n-1}(X^{n-1}, X^{n-2}) \cong \tilde{H}_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})$ by the natural factorization map. The diagram can be checked to be commutative, therefore we get that

$$\psi_{\beta *}\big(\mathrm{d}_{n-1}(e^n_\alpha)\big) = \psi_{\beta *}\Big(\big(\mathrm{d}_{n-1} \circ \Phi_{\alpha *}\big)(e^n_\alpha)\Big) = \Delta_{\alpha\beta *}(e^n_\alpha)$$

using the appropriate identifications. Since $\psi_{\beta *}$ is exactly the projection of X^{n-1}/X^{n-2} to ∂e^{n-1}_β we get that $\mathrm{d}_{\alpha\beta} e_\beta$ is nothing else but the just computed $\psi_{\beta *}\big(\mathrm{d}_{n-1}(e^n_\alpha)\big)$ hence $\mathrm{d}_{\alpha\beta}$ is the degree of the map

$$\mathbb{Z} \cong \tilde{H}_{n-1}(\partial e^n_\alpha) \to \tilde{H}_{n-1}(\partial e^{n-1}_\beta) \cong \mathbb{Z}$$

proving the statement. □

1.8 Manifolds

Definition 1.61. Suppose that M is a topological space with the following properties: T_2, M_2 and for every element $m \in M$ there exists an open neighborhood $U \ni x$ called a *chart* which is homeomorphic to $\mathbb{R}^n$ for some n. Such a space is called a *topological manifold.* The collection of neighborhoods homeomorphic to $\mathbb{R}^n$ form the set $\mathcal{A}$ and is called an atlas of M. (i.e. $\cup\mathcal{A} = M$). Similarly, one can define a *topological manifold with boundary* if the neighborhoods are homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$.

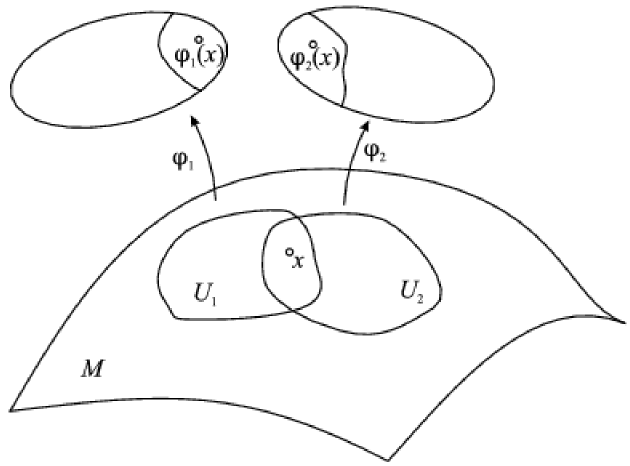


Figure 1.4: Transition maps

We usually assume that our manifold is connected so its dimension (i.e. the dimension of the charts) is unique since $\mathbb{R}^n$'s are not homeomorphic for different n's. For two overlapping charts, we can define the *transition maps* $g_{\alpha\beta} : \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \to \varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$ as $\varphi_\alpha \circ \varphi_\beta^{-1}$ where φ_α is the homeomorphism $U_\alpha \to \mathbb{R}^n$ (and similarly for β).

The general terminology is that a *closed manifold* stands for a manifold that is a compact and has no boundary, while the term '*compact manifold*' may allow a boundary.

Definition 1.62. A subset $\mathcal{A}_1 \subseteq \mathcal{A}$ is a *smooth* (or $\mathcal{C}^\infty$-) *structure* on a topological manifold of M if

- $\cup\mathcal{A}_1 = M$ i.e. they cover M.
- The transition functions $g_{\alpha\beta}$ are $\mathcal{C}^\infty$-function for all $U_\alpha, U_\beta \in \mathcal{A}_1$.

The $\mathcal{C}^\infty$-manifold structure is exactly the structure we need to define differentiability:

Definition 1.63. For a function $f : M \to \mathbb{R}$ we can take $f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ so it makes sense whether it is differentiable or not. Moreover, it does not depend on the choice of α since $g_{\alpha\beta}$ is differentiable and we can apply the chain rule on $f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$. So the definition is: f is *differentiable* if and only if it is differentiable in any (hence all) chart.

Remark 1.64. Note that although we have the notion of differentiability, we have no differential yet. So we still cannot "differentiate" because the result (the Jacobian) would still depend on the choice. (Fortunately, "$\mathrm{d}f = 0$" can be defined without any further choice, since the singularity of the Jacobian (taken in a char) is chart-invariant.

1.9 Orientation on manifolds

Reminder: An orientation in $\mathbb{R}^n$ corresponds to n basis vectors in a fixed order, and two such basises give the same orientation if the change of basis is given by a matrix of positive determinant. Now, we would like to rephrase this in terms of homology to (hopefully) get a generalization for other topological spaces, for example manifolds.

Notice that, $H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}; \mathbb{Z}) = \mathbb{Z}$ (clear after taking long exact sequence and realizing that $\mathbb{R}^n \backslash \{0\}$ is homotopy equivalent to $\mathbb{S}^n$). The choice of a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}; \mathbb{Z})$ orients $\mathbb{R}^n$.

Definition 1.65. A *local orientation* of a topological manifold M is a choice of a generator $\mu_n \in H_n(M, M\backslash\{m\}; \mathbb{Z})$ (where this homology is isomorphic to $\mathbb{Z}$ by Excision principle Theorem 1.46).

An *orientation* on M is a choice of local orientations $M \ni m \mapsto \mu_m$ such that for every $m \in M$ there exists a neighborhood $U \ni m$ and an element $\mu_U \in H_n(M, M\backslash U; \mathbb{Z})$ such that the embedding $(M, M\backslash U) \to (M, M\backslash\{p\})$ induces a map sending $\mu_U \mapsto \mu_p$ for all $p \in U$.

Remark 1.66. Orientations are introduced to give an alternative definition for the degree. Since there, we can take a regular value and count its preimages. However, we have to count with sign and that sign is given by the orientation of the function around the given preimage.

Tale: (About Morse theory) Consider a $\mathcal{C}^\infty$-manifold and a (generic enough) $\mathcal{C}^\infty$ function f on it. Consider the critical points (i.e. where $x \in M$ where locally $\mathrm{d}f = 0$). Now, consider the free Abelian group generated by the critical points, denoted by C. We can make it into a chain complex with the following boundary map: Fix a metric on M and consider the flows, i.e. some special kind of the diffeomorphisms of M such that the path of a point decreases f as fast as possible. These may bring critical points into other critical points. The coefficients in the boundary map is "how many flows are there between critical point $p \to q$ up to reparametrization". It is the Morse homology, but in fact it agrees with all the other homologies on sufficiently nice spaces.

Fact 1.67. *Not every manifold M is orientable but every manifold admits a double cover $\overline{M}$ which is orientable. Our goal is to construct $\overline{M}$.*

Definition 1.68. Let $\overline{M} := \{\mu_m \mid m \in M,\ \mu_m \text{ is a local orientation, that is a generator of } H_n(M, M\backslash\{m\}; \mathbb{Z}) \cong \mathbb{Z}\}$. Now, we specify the topology by its basis: for an open ball $B \subseteq \varphi_\alpha(\mathbb{R}^n) \subseteq M$ and $\mu_B \in H_n(M, M\backslash B)$ let $U(\mu_B) := \{\text{restrictions of } \mu_B \text{ to } M, M\backslash\{p\}\ \forall p \in B\}$. By this we get an $2:1$ continuous covering map $\overline{M} \to M$ by $\mu_m \mapsto m$.

Lemma 1.69. *M is orientable if and only if $\overline{M}$ has two components.*

Homework: (For 2nd of February)

- Show that $\overline{M}$ is orientable.
- Prove Lemma 1.69

Remark 1.70. Double covers of manifolds are in one-to-one correspondence with the index-2 subgroups of $\pi_1(M)$. Consequently, if $\pi_1(M)$ has no index two subgroups (e.g. $\pi_1(M) = 1$) then the only double cover is the trivial double cover hence a manifold with $\pi_1(M) = 1$ is orientable.

Example 1.71. Consider $\mathbb{C}P^n$. It has a CW decomposition by having one cell in every even dimension. Therefore, the resulting CW chain complex can have $\mathbb{Z}$'s in every even dimension and zeros elsewhere (Indeed, the boundary maps must be zeros everywhere). So the homologies are

$$H_i(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2k,\ 0 \le k \le n \\ 0 & \text{else} \end{cases}$$

Homework: (For 2nd of February) Compute $H_*(\mathbb{R}P^n; \mathbb{Z})$ and $H_*(\mathbb{R}P^n, \mathbb{Z}_2)$ using $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup D^n$.

Remark 1.72. There are spaces $\mathbb{C}P^\infty = \cup_{n \ge 1} \mathbb{C}P^n$ and $\mathbb{R}P^\infty = \cup_{n \ge 1} \mathbb{R}P^n$. These are factors of $\mathbb{S}^\infty \subseteq \mathbb{R}^\infty$ and $\mathbb{S}^\infty \subseteq \mathbb{C}^\infty$ in the following way: $\mathbb{R}P^\infty = \mathbb{S}^\infty/\mathbb{Z}_2$ and $\mathbb{C}P^\infty = \mathbb{S}^\infty/\mathbb{S}^1$.

Theorem 1.73. *A manifold M is orientable if and only if $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. (While $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is always true, without any assumption on orientation.) So an orientation can be seen as a choice of this isomorphism.*

Corollary 1.74. $\mathbb{R}P^n$

FIFTH LECTURE, 2ND OF FEBRUARY

1.10 Euler characteristic

Definition 1.75. Suppose that X is a finite CW-complex with c_i i-cells ($i \in \mathbb{N}$). Then $\chi(X) := \sum_{i=0}^{\infty}(-1)^i c_i \in \mathbb{Z}$ is called its *Euler-characteristic.*

Homework: (For 9th of February) If X is a finite CW-complex then

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i \mathrm{rk}\, H_i(X;\mathbb{Z}) = \sum_{i=0}^{\infty}(-1)^i \mathrm{rk}\, H_i(X;\mathbb{Q})$$

In particular, $\chi(X)$ is independent of the CW-decomposition, if X is a finite CW-complex. The appearing numbers $\mathrm{rk}\, H_i(X;\mathbb{Z})$ are called the *Betti numbers.*

Remark 1.76. For a finitely generated Abelian group G we define $\mathrm{rk}\, G = r$ if $G \cong \mathbb{Z}^r \oplus$ torsion where r is uniquely defined by the Fundamental Theorem of finitely generated Abelian groups. Observe that the above decomposition is not totally unique: the torsion part is unique as a subgroup but the free part is unique only as a quotient ring.

Here, we used that a finitely generated submodule is the direct sum of cyclics. The rings with this property are classified but sometimes we have to use such rings that does not satisfy it. For example, $\mathbb{F}[x, y]$. That is when algebraic geometry comes in the picture.

Definition 1.77. Suppose that M_1^n and M_2^n are two oriented topological manifolds. Consider $M_1 \backslash \nu(p_1)$ and $M_1 \backslash \nu(p_2)$ for some $p_i \in M_i$ ($i = 1, 2$) where ν stands for small enough neighborhood so that $\overline{\nu(p)} \cong \mathrm{int} D^n$ (together with orientation). Consider an orientation reversing homeomorphism $\varphi : \overline{\nu(p_1)} \to \overline{\nu(p_2)}$ and glue $M_1 \backslash \nu(p_1)$ to $M_2 \backslash \nu(p_2)$ along the boundaries of $\nu(p_i)$'s together using φ. That is, take

$$\big(M_1 \backslash \nu(p_1) \coprod M_2 \backslash \nu(p_2)\big) / (x \sim \varphi(x) \mid x \in \partial(M_1 \backslash \nu(p_1))$$

The result, denoted by $M_1 \# M_2$ is called the *connected sum.*

Remark 1.78.

1. The orientation-reversing thing is a corollary of the phenomenon "when you go out of one room, you go in to the other". So considering orientation-preserving maps would yield either a non-manifold or a non-orientable manifold depending on the case.

2. Note that the above construction is independent of p_1, p_2 and their neighborhoods as long as the manifolds are connected (and $\nu(p) \cong \mathrm{int} D^n$).

3. The same construction can be given for smooth (oriented) manifolds.

4. In the definition we purposefully used $D^n \to D^n$ maps since they are all isotopic (homotopic through homeomorphisms) while $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ maps does not always behave that nice, especially if we are in the smooth manifold situation.

2 Cohomology

Definition 2.1. Suppose that $(\oplus_n C_n, \partial)$ is a graded chain complex. The *dual (co)chain complex* is defined as

$$C^n := \mathrm{Hom}\,(C_n, \mathbb{Z}) = \{f : C_n \to \mathbb{Z} \mid f \text{ is a homomorphism}\}$$

with coboundary $\delta_n : C^n \to C^{n+1}$ given by the definition

$$C^{n+1} \ni (\delta_n f)(c) := f(\partial_n c) \in \mathbb{Z}$$

The resulting $(C^* = \oplus C^n, \delta = \oplus \delta_n)$ is a (co)chain complex. (We only need to check that $\delta^2 = 0$.) Now, we can take its cohomology:

$$H^n(C^*) := \mathrm{Ker}\,\delta_n / \mathrm{Im}\,\delta_{n-1}$$

Remark 2.2. The difference between chain and cochain complexes is manifested in the grading: in the case of chains the boundary is index-lowering but in the case of cochain complexes the coboundary is index-increasing. So algebraically there is no difference between the two notions.

Definition 2.3. Suppose that X is a topological space and apply this algebra to $(\oplus C_n(X), \partial = \oplus \partial_n)$ the singular chain complex associated to X. So we get a cochain complex $(C^*(X), \delta)$ and its cohomology $H^*(X; \mathbb{Z})$.

Questions: Does homology and cohomology determine each other? Or at least one another? Or what new do we get?

Theorem 2.4. (Universal Coefficient Theorem, for cohomologies) *Suppose that $C = \oplus C_n$ is a chain complex and G is an Abelian group. Consider the cochain complex given by the groups $C^n = \mathrm{Hom}\,(C_n, G)$. Then there exists a split short exact sequence:*

$$0 \longrightarrow \mathrm{Ext}\,(H_{n-1}(C), G) \longrightarrow H^n(C; G) \longrightarrow \mathrm{Hom}(H_n(C), G) \longrightarrow 0$$

where roughly speaking $\mathrm{Ext}\,(H_{n-1}(C); G)$ *is the group of Abelian extensions of G by $H_{n-1}(C)$.*

Proposition 2.5. Ext *has the following simple properties:*

1. $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$
2. *If H is free then* $\mathrm{Ext}(H, G) = 0$ *for all G.*
3. *If $H = \mathbb{Z}/n\mathbb{Z}$ then* $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) := G/nG$.

This is already enough to determine our Ext's since $H_{n-1}(C_n(X)) \cong \mathbb{Z}^r \oplus$ torsion so we can decompose the problem and determine the Ext's by 2) and 3). In particular, homologies determine cohomologies, even though they are not the same.

Observation: Suppose $f : C_* \to D_*$ is a chain map between chain complexes. Then f induces a map $f^\# : D^* \to C^*$ by $f^\#(\alpha)(c) = \alpha(f(c))$ for all $\alpha \in \mathrm{Hom}(D, \mathbb{Z})$ and $c \in C$ which is a cochain map. As such, it induces a map $f^* : H^*(D) \to H^*(C)$ sometimes denoted by $H^*(f)$.

Remark 2.6. Homology and cohomology are functors, i.e. $H_* : (X, Y) \mapsto H_*(X, Y)$ for all pairs of topological spaces, and similarly, a map $f : (X_1, Y_1) \to (X_2, Y_2)$ is mapped into $H_*(f) : H_*(X_1, Y_1) \to H_*(X_2, Y_2)$. This is what is called a *covariant functor*. Analogously, we can define $H^* : (X, Y) \mapsto H^*(X, Y)$ for all pairs of topological spaces and for a map $f : (X_1, Y_1) \to (X_2, Y_2)$ we can define its image $H^*(f) : H^*(X_2, Y_2) \to H^*(X_1, Y_1)$ where one should notice that the indices are in reversed order. That is why it is called a *contravariant functor*.

2.1 Cup product

Definition 2.7. Suppose that $\varphi \in C^k(X)$ and $\psi \in C^l(X)$. Define $\varphi \cup \psi \in C^{k+l}(X)$ as follows: for all $C_{k+l}(X) \ni \sigma : [v_0, \ldots, v_{k+l}] \to X$ define

$$(\varphi \cup \psi)(\sigma) := \varphi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,v_{k+l}]}) \in \mathbb{Z}$$

We would like to look at it as product of cohomology classes so some problems with well-definedness appears.

Lemma 2.8. $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$ *(In other words, we got a differential-graded algebra structure on $C^*(X)$)*

Proof. Suppose that σ is a singular $(k+l+1)$-chain $\sigma : \Delta^{k+l+1} \to X$. On this element, the right hand side is

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]})$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+l+1}]})$$

so adding these we get exactly the left hand side. □

Corollary 2.9. *If φ and ψ are cocycles then $\varphi \cup \psi$ is a cocycle. Moreover, if φ is a coboundary and ψ is a cocycle (or if φ is a cocycle and ψ is a coboundary) then $\varphi \cup \psi$ is a coboundary.*

Therefore, we get a map on the cohomologies: $\cup : H^k(X;\mathbb{Z}) \times H^l(X;\mathbb{Z}) \to H^{k+l}(X;\mathbb{Z})$.

Remark 2.10. There exists a $\mathbf{1} \in C^0(X;\mathbb{Z})$ such that for all $\sigma \in C_0(X;\mathbb{Z})$ we have $\mathbf{1}(\sigma) = 1 \in \mathbb{Z}$.

Corollary 2.11. *So $H^*(X;\mathbb{Z}) = \oplus_{n=0}^{\infty} H^n(X;\mathbb{Z})$ is a unital ring where $\mathbf{1}$ is the unit and the product is $\cup$. It is graded-commutative that is $x \cup y = (-1)^{kl} y \cup x$ for all $x \in H^k(X;\mathbb{Z})$ and $y \in H^l(X;\mathbb{Z})$.*

Remark 2.12. To define this ring structure, we indeed needed more than just the information in chain complexes since without the restriction of a simplex, we couldn't define the cup product.

Homework: (For 9th of February) Suppose that $f : X \to Y$ is a continuous map. Then $H^*(f) : H^*(Y;\mathbb{Z}) \to H^*(X;\mathbb{Z})$ is a ring homomorphisms.

Example 2.13. The following important cohomologies can be determined (as rings!):

1. $H^*(\mathbb{R}P^n;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ where x is the generator of $H^1(\mathbb{R}P^1;\mathbb{F}_2)$
2. $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[g]/(g^{n+1})$ where $g \in H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator (canonical by the choice of orientation given by the complex structure)
3. $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x]$
4. $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[g]$

2.2 Cap product

Definition 2.14. Assume that $k \geq l$. The *cap product* $\cap : C_k(X;\mathbb{Z}) \times C^l(X;\mathbb{Z}) \to C_{k-l}(X;\mathbb{Z})$ is defined as follows: let $\sigma \in C_k(X;\mathbb{Z})$ and $\varphi \in C_kC^l(X;\mathbb{Z})$ then its image is

$$\sigma \cap \psi := \varphi(\sigma|_{[v_0,\ldots,v_l]}) \cdot \sigma|_{[v_l,\ldots,v_k]}$$

yielding an element of $C_{k-l}(X;\mathbb{Z})$.

Proposition 2.15. $\partial(\sigma \cap \varphi) = (-1)^l \big(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi\big)$

Corollary 2.16. $\cap$ *gives a well-defined map* $H_k(X;\mathbb{Z}) \times H^l(X;\mathbb{Z}) \to H_{k-l}(X;\mathbb{Z})$.

Remark 2.17. Recall, if M is a closed topological manifold then orientability is equivalent to $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and a choice of such an isomorphism corresponds to an orientation, i.e. a choice of a fundamental cycle $[M] \in H_n(M;\mathbb{Z})$.

Theorem 2.18. (Poincaré duality) *Suppose that* M^n *is a closed topological manifold which is oriented with the fundamental class* $[M] \in H_n(M;\mathbb{Z})$. *Then the map* $PD : H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$ *mapping* $\alpha \mapsto [M] \cap \alpha$ *is an isomorphism.*

Remark 2.19. Recall that for closed, oriented surfaces $\pi_1(\Sigma_g)$ or $H_1(\Sigma_g)$ (which is the same as $H_*(\Sigma_g)$) gives a complete set of invariants. In dimension three, for a closed, oriented 3-manifold M, $\pi_1(M)$ (together with the orientation) is a "complete" set of invariants. In dimension four: $\pi_1(M)$ is too complicated (every finitely presented group can appear, and those are not algorithmically classifiable). Consider simply connected, oriented smooth 4-manifolds X_1 and X_2. Then they are homeomorphic if and only if $H^*(X_1;\mathbb{Z})$ and $H^*(X_2;\mathbb{Z})$ are isomorphic as rings. Notice that these are smooth manifold classified up to homeomorphism (not up to diffeomorphism!).

Example 2.20. The fact that the ring structure is indeed needed is shown by the example $\mathbb{S}^2 \times \mathbb{S}^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Their cohomologies are

$$H^0(X;\mathbb{Z}) = \mathbb{Z} \qquad H^1(X;\mathbb{Z}) = 0 \qquad H^2(X;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \qquad H^3(X;\mathbb{Z}) = 0 \qquad H^4(X;\mathbb{Z}) = \mathbb{Z}$$

but their cup product structure $\cup : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z})$ in the case of $\mathbb{S}^2 \times \mathbb{S}^2$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where the (i,j)-element means the coefficient of $[X]$ in the product of e_i and e_j. While the cup product in the case of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the same way as before. These are genuinely different, since in the first case, every square is in $2\mathbb{Z}$ but in the second case it is not true.

Sixth Lecture, 26th of February

3 Obstruction Theory

Motivation: After the last lecture, one can have the feeling that cohomology is "homology + more algebra". However, in fact it can have a lot more application since many obstructions to certain phenomenons or constructions are realized as elements in cohomology.

3.1 Homotopy groups

Homotopy groups are the generalizations of the fundamental group (the group of loops starting and ending at a fixed point x_0 up to x_0-fixing homotopy). Poincaré has invented them to prove that there exists a 3-manifold with all vanishing homologies but not homeomorphic to $\mathbb{S}^3$.

Definition 3.1. Let $n \geq 1$ and

$$\pi_n(X, x_0) := \{f : (I^n, \partial I^n) \to (X, x_0)\}/\text{homotopy}$$

where the homotopy of pairs of spaces means a homotopy that brings ∂I^n into x_0 at every "level" of the homotopy.

Example 3.2. In $n = 1$ we have I^1 the interval but when we map the two boundary points of it, then its image is a loop in the space. Similarly, in $n = 2$ it gives a spheroid.

This $\pi_n(X, x_0)$ admits a group structure:

$$(f+g)(s_1, \dots, s_n) := \begin{cases} f(2s_1, s_2 \dots, s_n) & \text{if } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \dots, s_n) & \text{if } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

in pictures it "puts the two maps next to each other", contracting them half in one direction.

Proposition 3.3. *This operation is a group operation.*

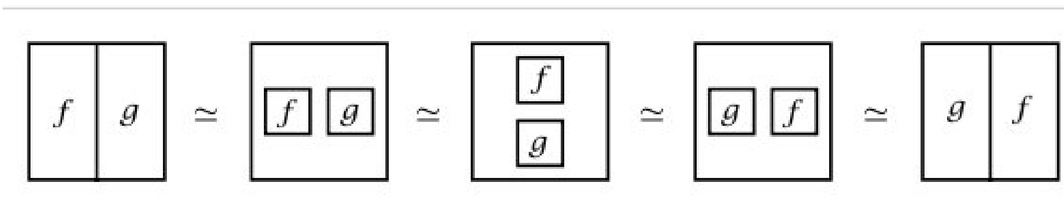
Proof. We only give the definition of the inverse, then the statement becomes a straightforward check.

$$(-f)(s_1, \dots, s_n) := f(1 - s_1, s_2, \dots, s_n)$$

so we are doing f "backwards" using the first direction. This is basically the same as the inverse in the case of $\pi_1(X, x_0)$, without modifying the maps in the other coordinates. $\square$

Proposition 3.4. *$\pi_n(X, x_0)$ is a commutative once $n \geq 2$.*

Proof. The proof is pictorial.



It is based on the fact that I^n for $n \geq 2$ has a one-parameter family of diffeomorphisms such that it switches the two halves of the cube. $\square$

Remark 3.5. $\pi_n(X, x_0)$ can be very complicated even for nicely looking spaces. Take $\mathbb{S}^1 \vee \mathbb{S}^2$ then we get $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2, s_0) = \mathbb{Z}^\infty$ since when mapping $\mathbb{S}^2 \cong \mathbb{S}^2 \vee I^1$ into itself then we can choose a degree on the $\mathbb{S}^2 \to \mathbb{S}^2$ part and a degree on the $I^1 \to \mathbb{S}^1$ part. This argument is not completely rigorous but it can be made into one.

Lemma 3.6. *Suppose that $p : \tilde{X} \to X$ is a covering. Then*

1. *$\pi_1(\tilde{X})$ injects into $\pi_1(X)$. (Moreover, its index is the degree of the covering.)*
2. *$p_* : \pi_n(\tilde{X}) \to \pi_n(X)$ is an isomorphism for $n \geq 2$.*

Proof. The first is known by generalities on the fundamental group. The second follows by the lifting property: a map $(\mathbb{S}^n, s_0) \to (X, x_0)$ can be lifted into a map $(\mathbb{S}^n, s_0) \to (\tilde{X}, x_0)$ if and only if $\pi_1(\mathbb{S}^n, s_0)$ maps into the subgroup $\pi_1(\tilde{X}, x_0) \leq \pi_1(X, x_0)$. However, $\pi_1(\mathbb{S}^n, s_0) = 0$ so the condition is automatically satisfied. $\square$

Remark 3.7. In the previous example the simply connected cover is a $\mathbb{Z}$-many spheres wedged to the integers on $\mathbb{R}^1$ so we can now see the reason of the complicated behavior of $\pi_n(X) \cong \pi_n(\tilde{X})$.

Remark 3.8. The π_k's of $\mathbb{S}^n$ is not known for all k, n-s. What is more or less known and is interesting is the homotopy groups of $O(n)$ and $U(n)$. We will discuss that later, the important "tool" here is Bott's periodicity.

Notice that $\pi_n(X, x_0)$ depends on the choice of x_0. If X is (path-wise) connected then $\pi_n(X, x_0) \cong \pi_n(X, y_0)$ but the isomorphism is not canonical, it can be given by choosing a path γ from x_0 to y_0. So we cannot talk about elements of $\pi_n(X)$. This argument gives that $\pi_1(X, x_0)$ acts on $\pi_n(X, x)$, i.e. we get a map $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_n(X, x_0))$. Sometimes it can happen that this map is trivial.

Definition 3.9. X is *n-simple* if $\pi_1(X, x_0)$ acts trivially on $\pi_n(X, x_0)$.

E.g. this happens if $\pi_1(X, x_0) = 1$.

3.2 Relative homotopy groups

Suppose that (X, A, x_0) is a triple where $x_0 \in A \subseteq X$. Then we can define the following:

Definition 3.10. The n-th *relative homotopy group* of (X, A, x_0) is

$$\pi_n(X, A, x_0) := \{f : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)\}/\text{homotopy}$$

where $I^n \supseteq I^{n-1} := \{x \in I^n \mid s_n = 0\}$ i.e. it is the "front face" of I^n and $J^{n-1} = \partial I^n \backslash I^{n-1}$.

Proposition 3.11. *There exists a long exact sequence:*

$$\ldots \longrightarrow \pi_n(A, x_0) \xrightarrow{\text{emb}_1} \pi_n(X, x_0) \xrightarrow{\text{emb}_2} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, x_0) \longrightarrow \ldots$$

where emb_1 *is the induced map by* $(A, x_0) \to (X, x_0)$*,* emb_2 *is the induced map by* $(X, x_0, x_0) \to (X, A, x_0)$ *and* δ *is the restriction to the boundary.*

Proposition 3.12. *If* $f : (X, x_0) \to (Y, y_0)$ *is a homotopy equivalence then* $f_{*n} : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ *is an isomorphism for all* n*. (Straightforward.)*

Theorem 3.13. (Whitehead) *Suppose that* X *and* Y *are CW-complexes and* $f : (X, x_0) \to (Y, y_0)$ *is a map such that* f_* *is an isomorphism for all* n*. Then* f *is a homotopy equivalence.*

Disclaimer: The isomorphism of the homotopy groups does not necessarily mean that the spaces are homotopy equivalence, only if there is a single map that induces this isomorphism on every level, simultaneously.

3.3 Connection between π_n and H_n.

Theorem 3.14. (Hurewitz) *Suppose that* $\pi_i(X) = 0$ *for all* $i < n$ *and* $n \geq 2$*. Then* $\tilde{H}_i(X, \mathbb{Z}) = 0$ *for all* $i < n$ *and* $H_n(X; \mathbb{Z}) \cong \pi_n(X)$*.*

Definition 3.15. X is a $K(G, n)$-space (for an Abelian group G) if $\pi_i(X) = 0$ if $i \neq n$ and $\pi_n(X) \cong G$.

Example 3.16. $K(\mathbb{Z}, 1) = \mathbb{S}^1$ and $K(\mathbb{Z}_2, 1) \cong \mathbb{R}P^\infty$ by covering spaces.

Problem: Find continuous maps $K \to X$.

As it is stated, it is not too interesting, the constant maps are always there. Let's refine it as follows: Suppose that K is a CW-complex with $(n-1)$-skeleton denoted by K^{n-1} and suppose that $f : K^{n-1} \to X$ is already given. The problem is whether we can extend f to K^n or not.

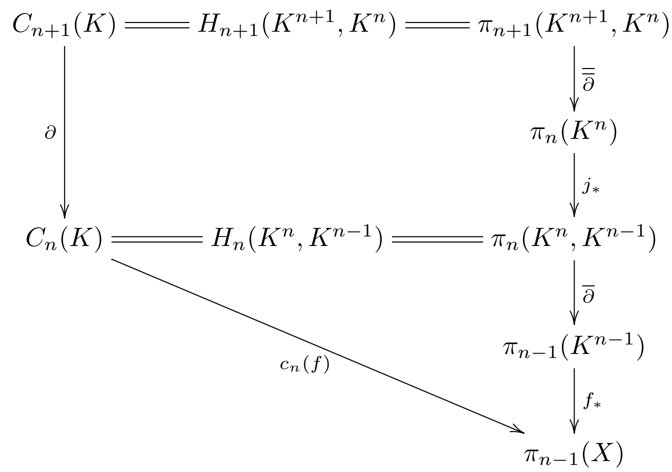
Assumption: From now on, assume that X is n-simple (e.g. $\pi_1(X) = 0$).

Consider $K^n = K^{n-1} \cup \bigcup_\alpha e^n_\alpha$ where f is defined on K^{n-1}. We know that $e^n_\alpha \cong D^n$ with $\partial D^n \cong \mathbb{S}^{n-1}$ on which f is already defined, but the problem would be (without the assumption of n-simple) that we have no base point in X. So consider $[f]_\alpha \in \pi_{n-1}(X)$ as the spheroid $f(\partial e^n_\alpha)$ which is now well defined since it does not matter into which point f brings the base point of K. This gives a map $e^n_\alpha \to [f]_\alpha$ which is, putting it together for all α's, a cochain in $C^n(K, \pi_{n-1}(X)) = \mathrm{Hom}(C_n(K^n), \pi_{n-1}(X))$. So, at the end, we got an element $c_n(f) \in C^n(K^n, \pi_{n-1}(X))$.

Proposition 3.17. (without proof, easy) f *extends to* K^n *if and only if* $c_n(f) = 0$.

Theorem 3.18. *The cochain* $c_n(f) \in C^n(K^n, \pi_{n-1}(X)$ *is always a cocycle, that is* $\delta(c_n(f)) = 0$.

Proof. Observe that $\delta(c_n(f)) \in \mathcal{C}^{n+1}(K, \pi_{n-1}(X))$ so we can take the diagram



where we used the Hurewitz theorems at the equalities. Moreover, $\bar{\partial}$, $\bar{\bar{\partial}}$ and j_* are coming from the appropriate the long exact sequences given by Proposition 3.11. The diagram is commutative hence we can see what is $\delta(c_n(f))$:

$$\big(\delta c_n(f)\big)(\sigma) = c_n(f)\big(\partial\sigma\big) = f_*(\overline{\partial}(j_*(\overline{\overline{\partial}}\sigma)))$$

Here, we can apply Proposition 3.11 about the long exact sequences of homotopy groups so $\pi_n(K^n) \xrightarrow{j_*} \pi_n(K^n, K^{n-1}) \xrightarrow{\overline{\partial}} \pi_{n-1}(K^{n-1})$ is exact and the above composition is necessarily zero. □

Definition 3.19. We can define $C_n(f) = [c_n(f)] \in H^n(K, \pi_n(X))$ i.e. the homotopy class of $c_n(f)$. This will be some kind of obstruction class but it sheds a different light on Proposition 3.17 that did not take cohomology class.

Question: What is the meaning of $C_n(f)$?

Recall that by Proposition 3.17, f cannot be extended if $C_n(f) \neq 0$. But what happens when $c_n(f)$ is a coboundary?

Theorem 3.20. (Fuchs - Fomenko - Gutenmacher, page 114-116) $C_n(f) \in H^n(K; \pi_{n-1}(X))$ *is zero if and only if there exists a* $g : K^{n-1} \to X$ *such that* $f|_{K^{n-2}} \equiv g|_{K^{n-2}}$ *and* g *extends to* K^n.

Suppose that $f, g : K^{n-1} \to X$ are two maps with $f|_{K^{n-2}} \equiv g|_{K^{n-2}}$.

Definition 3.21. The *difference cochain* $\mathrm{d}^{n-1}_{f,g} \in C^{n-1}(K, \pi_{n-1}(X))$ is defined as follows: Consider an $(n-1)$-cell e^{n-1}_α in K^{n-1}. Since $\partial e^{n-1}_\alpha \subseteq K^{n-2}$ clearly $f = g$ on ∂e^{n-1}_α. A map $\mathbb{S}^{n-1} \to X$ can be defined by taking $f|_{e^{n-1}_\alpha}$ on the upper hemisphere and $g|_{e^{n-1}_\alpha}$ on the lower hemisphere. By this $\mathrm{d}^{n-1}_{f,g}(e^{n-1}_\alpha)$ is defined as an element $[\mathbb{S}^{n-1} \to X] \in \pi_{n-1}(X)$. So $\mathrm{d}^{n-1}_{f,g}$ becomes a $\mathcal{C}_{n-1}(K) \to \pi_{n-1}(X)$ map.

Notice that $\mathrm{d}^{n-1}_{f,g} = 0$ means that f and g are homotopic on K^{n-1}. Indeed, $\mathrm{d}^{n-1}_{f,g} \equiv 0 \in \pi_{n-1}(X)$ means that for all cell e^{n-1}_α the above defined maps are null-homotopic. Moreover, $\delta\mathrm{d}^{n-1}_{f,g} = 0$ would mean that $\mathrm{d}^{n-1}_{f,g}$ is a cycle.

Claim 3.22. $\delta\mathrm{d}^{n-1}_{f,g} = c_n(f) - c_n(g)$

Proof. Let's evaluate it on a $\sigma \in C_n(K)$:

$$\delta \mathrm{d}_{f,g}^{n-1}(\sigma) = \mathrm{d}_{f,g}^{n-1}(\partial\sigma) = \mathrm{d}_{f,g}^{n-1}\Big(\sum_\alpha \left[\sigma^n : \sigma_\alpha^{n-1}\right]\sigma_\alpha^{n-1}\Big) = \sum_\alpha \left[\sigma^n : \sigma_\alpha^{n-1}\right]\mathrm{d}_{f,g}^{n-1}\left(\sigma_\alpha^{n-1}\right)$$

So we only have to check that $\big(c_n(f) - c_n(g)\big)(\sigma)$ is the same as this.

Homework: (For 23th of February) Compute the value of $\big(c_n(f) - c_n(g)\big)(\sigma)$

□

Lemma 3.23. *Suppose that $f : K^{n-1} \to X$ is given and $\mathrm{d} \in C^{n-1}\big(K, \pi_{n-1}(X)\big)$ is a given cochain. There exists a map $g : K^{n-1} \to X$ such that $\mathrm{d}_{f,g}^{n-1} = \mathrm{d}$ and $f|_{K^{n-2}} = g|_{K^{n-2}}$.*

Proof. Define $g|_{K^{n-2}} := f|_{K^{n-2}}$ and then consider an $(n-1)$-cell e_α^{n-1} in K^{n-1}. Take $f(e_\alpha^{n-1})$ and consider $\mathrm{d}(e_\alpha^{n-1}) \in \pi_{n-1}(X)$, so it can be represented by a map $\varphi : \mathbb{S}^{n-1} \to X$. So we can define "$g = f + d$" i.e. take the "connected sum" of φ and $f|_{e_\alpha^{n-1}}$ where one can easily work out what the connected sum of two maps into a connected space means.

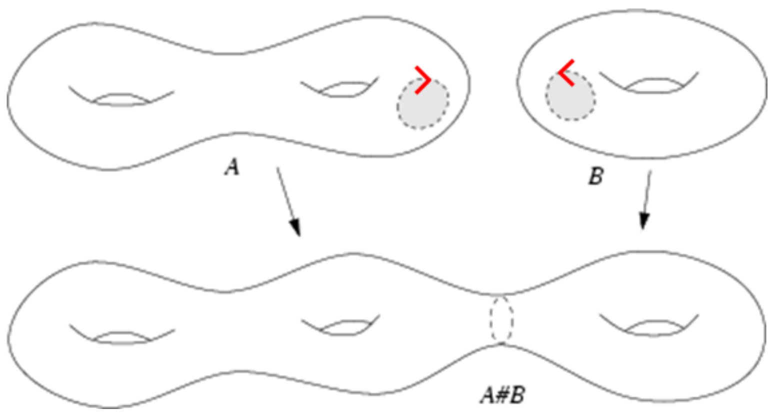


Figure 3.1: Connected sum

□

Proof. (of Theorem 3.20:) First, we prove direction $\Leftarrow$: Suppose that g is as stated. This means that $c_n(g) = 0$ by Proposition 3.17. Then by Claim 3.22, it means that

$$\delta \mathrm{d}_{f,g}^{n-1} = c_n(f) - c_n(g) = c_n(f)$$

so $c_n(f)$ is a coboundary. Therefore, $C_n(f) = [c_n(f)] = 0$. (Notice that this part of the proof is constructive in some sense.)

For direction $\Rightarrow$ we will use the lemma: If $C_n(f) = 0$ in $H^n(K, \pi_{n-1}(X))$ i.e. there exists $\mathrm{d} \in C^{n-1}(K, \pi_{n-1}(X))$ such that $\delta\mathrm{d} = c_n(f)$. However, by Lemma 3.23 we know that there is a g such that $\mathrm{d} = \mathrm{d}_{f,g}^{n-1}$. So take this g and "it will be OK for the theorem". We know that g is the same as f on K^{n-2}, now we only have to check that $c_n(g) = 0$ so then – again, by Proposition 3.17 – it can be extended. So let's compute:

$$c_n(f) = \delta\mathrm{d} = \delta\mathrm{d}_{f,g}^{n-1} = c_n(f) - c_n(g)$$

Hence we are done. □

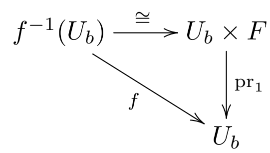
Seventh Lecture, 23th of February

Remark 3.24. About the obstructions c_n:

- If f and g are homotopic then $c_n(f) = c_n(g)$.
- If f, g and h are equal on K^{n-2} where $f, g, h : K^{n-1} \to X$ then the cocycle condition $\mathrm{d}^{n-1}_{f,g} + \mathrm{d}^{n-1}_{g,h} = \mathrm{d}^{n-1}_{f,g}$ and $\mathrm{d}^{n-1}_{f,g} = -\mathrm{d}^{n-1}_{g,f}$ holds.

4 Fiber bundles

Definition 4.1. A continuous function $f : E \to B$ is called a *fiber bundle* if it is locally trivial that is, for every point $b \in B$ there exists a neighborhood $U_b \ni b$ such that the following diagram commutes for some fixed topological space F:



where $\cong$ means homeomorphic. Informally, E is some kind of a twisted product: it is locally a direct product of B and F but globally it not necessarily is. Formally, it is the sixtuple $\big(E, B, F, f, (U_b)_{b \in B}, (\cong_{U_b})_{b \in B}\big)$, which does not say anything about its "semi-direct product" nature.

Example 4.2. Smallest but "main" example: $E =$ Möbius band.

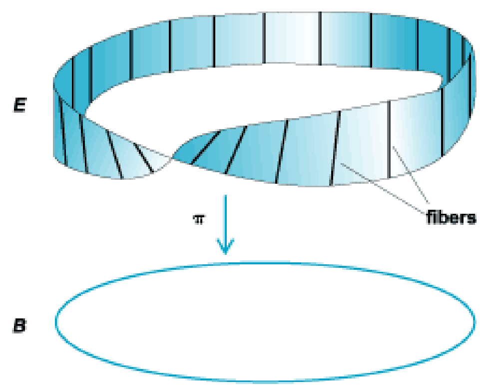


Figure 4.1: Möbius band as a fiber bundle

It is a fiber bundle over $\mathbb{S}^1$ and $I^1 = [0, 1]$ which is clearly nontrivial since $\mathbb{S}^1 \times I^1$ has non-connected boundary while the boundary of the Möbius band is connected.

Remark 4.3. Suppose that U_α and $U_\beta \subseteq B$ are two trivializing neighborhoods. Then $f^{-1}(U_\alpha \cap U_\beta)$ is homeomorphic to $(U_\alpha \cap U_\beta) \times F$ (which lies in $U_\alpha \times F$ and $U_b \times F$) in two different ways. This gives us a map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$$

where $\mathrm{Homeo}(F)$ stands for all the homeomorphisms of F.

Proposition 4.4. *Properties of g:*

- $g_{\alpha\alpha} = \mathrm{id}_F$
- $g_{\beta\alpha} = g^{-1}_{\alpha\beta}$ *where* $^{-1}$ *stands for the inverse of* $\mathrm{Homeo}(F)$*, so we invert* $g_{\alpha\beta}$ *"pointwise" not as an* $U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$ *function.*
- $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{Id}_F$ *on* $U_\alpha \cap U_\beta \cap U_\gamma$ *where* $\cdot$ *is the multiplication of* F*, not the composition of* g*'s.*

Definition 4.5. $(\{U_\alpha\}, \{g_{\alpha\beta}\})$ associated to a bundle $E \to B$ is the *cocycle structure* of the bundle.

Proposition 4.6. *The cocycle structure determines the bundle.*

Proof. Given a $U_\alpha \times F$ we know how we should identify it with $U_\beta \times F$: by $g_{\alpha\beta}$. So, formally

$$E := \coprod \Big(U_\alpha \times F\Big) \Big/ \{g_{\alpha\beta}\}_{\alpha\beta}$$

i.e. we identify the points of $U_\alpha \times F$ and $U_\beta \times F$ and we take factor-topology. The properties mentioned in Proposition 4.4 are exactly the requirements to make this work. The projection $f : E \to B$ can be defined in an obvious way. □

Definition 4.7. Suppose that $G \leq \text{Homeo}(F)$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ (instead of $\text{Homeo}(F)$) then $E \to B$ is called a *G-bundle*. It G preserves some extra structure on F then we may make sense of this structure on E as well.

4.1 G-bundles

An important subclass of bundles when $F = \mathbb{R}^n$. Here, we would like to add the vectors of the fiber F but the result of addition may depend on the chart U_α. To solve this, assume that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ so addition and multiplication by a scalar becomes well-defined.

Definition 4.8. A $GL_n(\mathbb{R})$-bundle with $F = \mathbb{R}^n$ is called a *vector bundle*. If $n = 1$ then it is called a *line bundle*.

Remark 4.9. We may go further: we could restrict ourselves to $G = O(n) \subseteq GL_n(\mathbb{R})$ so we can assume that $g_{\alpha\beta}$ keeps length and angle as well. Then we can make sense of length and angle of the elements of E. Similarly, we may assume $G = SO(n)$. If we go too far, i.e. $G = \{1\}$ then this will result that $E \to B$ becomes trivial.

The usefulness of this restriction is based on the fact the $\text{Homeo}(F)$ is really a huge group in some sense.

Question: How do we build up a G-bundle with fiber F? Most of the time, this G will be a (fixed) finite-dimensional Lie group.

Answer: It requires two components:

- A cocycle structure $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$
- And a representation $\lambda : G \to \text{Aut}(F)$ where $\text{Aut}(F)$ is the set of homeomorphisms of F respecting its extra structure, no matter what this extra structure is.

Definition 4.10. Not that G admits a representation $G \to \text{Aut}(G)$ by right multiplication $g \mapsto (h \mapsto h \cdot g)$. This leads to the concept of a *principal bundle*: it can have arbitrary cocycle structure but with the (right) regular representation of G.

A principal bundle is a "bundle of affine Lie groups", i.e. the fibers are $\text{Aut}(G)$ but without a fixed unit element.

Definition 4.11. Suppose that $f : E \to B$ is a bundle. A map $\sigma : B \to E$ is called a *section* if $f \circ \sigma = \text{id}_B$. In other words, $\sigma(x) \in E_x := f^{-1}(x)$ for all $x \in B$.

Homework: (For 2nd of March): A principal bundle admits a section if and only if it is trivial. (This is not true for arbitrary G-bundles: e.g. the conjugation action $G \to \text{Aut}(G)$: $g \mapsto (h \mapsto g^{-1}hg)$ always have a section but it is not always trivial.)

Suppose that $h : P \to B$ is a principal G-bundle and $\lambda : G \to \text{Aut}(F)$ is a representation of G. We can construct the associated fiber bundle $E = P \times_\lambda F$ with fiber F as follows

$$E := P \times F / \sim$$

where $(pg, f) \sim (p, \lambda(g) \cdot f)$. This construction automatically comes with a projection $\pi : P \times_\lambda F \to B$ as $\pi(p, f) = h(p)$.

4.2 Vector bundles

Let $P \to B$ be a principal $GL_n(\mathbb{R})$-bundle and $\lambda : GL_n(\mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^n) = GL_n(\mathbb{R})$ a representation. Then $E = P \times_\lambda \mathbb{R}^n \to B$ is a fiber bundle with $\mathbb{R}^n$.

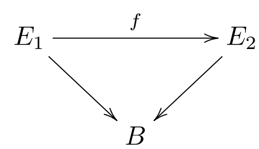
These bundles always have a section: since every E_x has a unique element, namely the zero vector. This section is called the zero section.

Definition 4.12. In the vector bundle case, not only a zero section exists but we can add the elements of the *space of section* and multiply them with scalars so

$$\Gamma(E) := \{\sigma : B \to E \mid \sigma \text{ section of the fiber bundle } E \to B\}$$

becomes a vector space (mostly infinite dimensional).

Definition 4.13. Two fiber bundles $E_1 \to B$ and $E_2 \to B$ are called *equivalent* if there exists a homeomorphism f making the following diagram commutative:



If E_1 and E_2 are G-bundles, the φ should respect this extra structure.

Assumption: From now on $E \to B$ with fiber $\mathbb{R}^n$ will be a vector bundle.

Remark 4.14. There is no reason to stick to $\mathbb{R}^n$ vector bundles with fiber $\mathbb{C}^n$ are also important objects. (Note, however, that no holomorphic structure is assumed on them at this topic.)

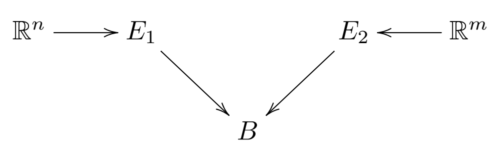
Definition 4.15. Suppose that $\pi : E \to B$ is a fiber bundle and let $f : B_1 \to B$ a continuous map. Then E can be *pulled back* to a fiber bundle over B_1:

$$f^*E := \{(e, b) \in E \times B_1 \mid \pi(e) = f(b)\}$$

This will admit a projection $f^*E \to B_1$ by $(e, b) \mapsto b$. Alternatively, it is called the *fibred product* of E and B_1 over B.

On the level of cocycle structures we pull back $g_{\alpha\beta}$'s on U_α's as $g_{\alpha\beta} \circ f$ on $f^{-1}(U_\alpha)$.

Suppose that E_1 and E_2 are vector bundles over the same space B:



Then we can define their direct sum: $E_1 \oplus E_2$ by

$$g^{\oplus}_{\alpha\beta} = \begin{pmatrix} g^1_{\alpha\beta} & 0 \\ 0 & g^2_{\alpha\beta} \end{pmatrix}$$

where g^1 and g^2 are the cocycle structures of E_1 and E_2.

Similarly, one can define $E_1 \otimes E_2$ by $g^{\otimes}_{\alpha\beta} = g^1_{\alpha\beta} \otimes g^2_{\alpha\beta}$.

Remark 4.16. If E_1 and E_2 are line bundles then $E_1 \otimes E_2$ is also a line bundle with $g^{\otimes}_{\alpha\beta} = g^1_{\alpha\beta} \cdot g^2_{\alpha\beta} \in \mathbb{R}$. Moreover, if L is a line bundle then one can consider L^{-1} by taking $g^{-1}_{\alpha\beta}$. This is not the case in higher dimension because of noncommutativity.

Proposition 4.17. *The set of equivalence classes of line bundles (either complex or real) forms an Abelian group with* $\otimes$.

Definition 4.18. If $E \to B$ is an n-dimensional vector bundle then one can consider $\det E \to B$ the *determinant line bundle* defined by $\det g_{\alpha\beta}$'s.

Homework (For 2nd of March):

1. Suppose that $E \to B$ is an $O(n)$-bundle. Then its cocycle structure can be reduced to $SO(n)$ if and only if $\det E$ is trivial.
2. Consider the set of principal $\mathbb{Z}/2\mathbb{Z}$-bundles over a fixed CW-complex B. There exists a bijection between this set and the homeomorphism-classes of B and similarly a bijection with the subgroups of $\pi_1(B)$ with index two and a bijection with the elements of $H^1(B;\mathbb{Z}/2\mathbb{Z})$.

4.3 Milnor construction

Let G be a (compact) Lie group. Let $E_G = G * G * \cdots * G * \dots$ where this means

$$E_G := \left\{ \langle t, x\rangle = \langle t_0x_0, t_1x_1, t_2x_2, \dots\rangle \mid x_i \in G,\ t_i \in [0,1],\ \text{only finitely many } t_i \neq 0,\ \sum t_i = 1 \right\}$$

It admits many maps $t_i : E_G \to [0,1]$ mapping $\langle t, x\rangle$ into t_i. On $t_i^{-1}(0,1]$ there is a map $x_i : t_i^{-1}[0,1] \to G$. The topology of E_G is the weakest that makes these maps continuous. The group G acts on E_g by $\langle x, t\rangle y = \langle xy, t\rangle$ for all $y \in G$.

Theorem 4.19.

- *$E_G \to E_G/G =: B_G$ is a principal G-bundle.*
- *This is a universal G-bundle, that is, for every principal G-bundle over paracompact spaces (e.g B can be a compact space or a manifold) there is a function $f : B \to B_G$ such that $f^*E_G = E$. Moreover, if $f_1, f_2 : B \to B_G$ has $f_1^*E_G \cong f_2^*E_G$ then f_1 and f_2 are homotopic.*

Example 4.20. If $G = \mathbb{Z}_2$ then $E_{\mathbb{Z}_2}$ can be chosen to be $\mathbb{S}^\infty$ and then $B_{\mathbb{Z}_2} = \mathbb{R}P^\infty$. This is the generalization of the boundary of the Möbius bundle $\mathbb{S}^1 \to \mathbb{R}P^1$ or $\mathbb{S}^3 \to \mathbb{R}P^3$ the $SU(2) \to SO(3)$ bundle.

If $G = \mathbb{S}^1 = U(1) = SO(2)$ then $E_{\mathbb{S}^1}$ can be chosen to $\mathbb{S}^\infty$ with $B_{\mathbb{S}^1} = \mathbb{C}P^\infty$. This is the generalization of the Hopf fibration $\mathbb{S}^3 \to \mathbb{C}P^1$.

Remark 4.21. Recall that $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[x]$ where $0 \neq x \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[g]$ for a generator $g \in H^2(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

Theorem 4.22. [reformulation of the main part of 4.19] *Principal G-bundles over B (up to equivalence) are in bijective correspondence with $[B, B_G]$ i.e. the maps $B \to B_G$ up to homotopy.*

4.4 Tangent bundle

Definition 4.23. Suppose that M is a smooth manifold. Then there exists a special bundle associated to M: the *tangent bundle* $TM \to M$. For this, $\{U_\alpha\}$'s should be manifold charts and the transition function are $\mathcal{C}^\infty$-functions.

A tangent vector at $p \in M$ is a map $v : \mathcal{C}^\infty(M) \to \mathbb{R}$ is a linear map satisfying Leibniz's rule:

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g)$$

In a coordinate chart, such a v can be expressed as a linear combination $v = \sum \alpha_i \frac{\partial}{\partial x_i}$ for some $\alpha_i \in \mathbb{R}$. This shows that for every p the space of such v's constitute an n-dimensional vector space where $n = \dim M$.

More generally, a vector field can be defined as a map $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ satisfying Leibniz's rule:

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$$

Now, the cocycle structure of the tangent bundle can be defined as $g_{xy} = \left(\frac{\partial y_i}{\partial x_j}\right)_{i,j}$ on $U_x \cap U_y$ for the neighborhoods $U_x \ni x$ and $U_y \ni y$.

5 Characteristic classes

Notice that a map $f : X \to \mathbb{R}P^\infty$ gives rise to an element $H^1(X;\mathbb{Z}_2)$. Therefore,

$$f^* : H^1(\mathbb{R}P^\infty;\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2)$$

$$0 \neq x \mapsto f^*x$$

So we got a map $[X, \mathbb{R}P^\infty] \to H^1(X;\mathbb{Z}_2)$.

Similarly, for $\mathbb{C}P^\infty$ a map $F : X \to \mathbb{C}P^\infty$ gives rise to an element $F^*(g) \in H^2(X;\mathbb{Z})$ for the fixed generator $g \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ which is the unique g such that $g([\mathbb{C}P^1]) = 1$ for $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^\infty;\mathbb{Z})$.

Theorem 5.1. *(without proof) These maps are bijections.*

Remark 5.2. More generally, for $K(G,n)$ (see Definition 3.15) we have $H^n(X;G) = [X, K(G,n)]$.

Recall that the principal $O(1) = \mathbb{Z}_2$-bundles over B are in bijection with $[B;\mathbb{R}P^\infty] = H^1(B;\mathbb{Z}_2)$.Similarly, the principal $U(1)$-bundles over B are in bijection with $[B;\mathbb{C}P^\infty] \to H^2(B;\mathbb{Z})$. However, here we have a group structure on the principal bundles and on the cohomologies as well.

Claim 5.3. (without proof) These are isomorphisms of groups. The case of $O(1)$ is called Stiefel-Whitney isomorphism and the second one is called Chern isomorphism.

Eighth Lecture, 2nd of March

Collecting the mentioned isomorphisms, we get the following:

Theorem 5.4. $H^2(X;\mathbb{Z}) \cong [X, \mathbb{C}P^\infty]$ *where the map* $H^2(X;\mathbb{Z}) \leftarrow [X, \mathbb{C}P^\infty]$ *is defined in the obvious way:*

$$[X, \mathbb{C}P^\infty] \ni f \mapsto f^*(g) \in H^2(X;\mathbb{Z})$$

and similarly, $H^1(X;\mathbb{Z}_2) \cong [X, \mathbb{R}P^\infty]$. *Moreover, these spaces are in* $1-1$ *correspondence with the following:*

$$H^2(X;\mathbb{Z}) \cong [X, \mathbb{C}P^\infty] \leftrightarrow \{\text{principal } \mathbb{S}^1 \text{ -bundles on } X\ \} \leftrightarrow \{\text{complex line bundles on } X\}$$

which map $\{\text{complex line bundles on } X\} \to H^2(X;\mathbb{Z})$ *is a group homomorphism, called the first Chern class. And similarly,*

$$H^1(X;\mathbb{Z}_2) \cong [X, \mathbb{R}P^\infty] \leftrightarrow \{\text{principal } \mathbb{Z}_2\text{-bundles on } X\ \} \leftrightarrow \{\text{real line bundles on } X\}$$

which map $\{\text{real line bundles on } X\} \to H^1(X;\mathbb{Z}_2)$ *is a group homomorphism, called the first Stiefel-Whitney class.*

5.1 Tautological bundles

Definition 5.5.

$$\tau = \left\{(\ell, u) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid u \in \ell\right\} \subseteq \mathbb{C}P^n \times \mathbb{C}^{n+1}$$

$$\mathbb{C}P^n \xleftarrow{\ \mathrm{pr}_1\ } \tau \xrightarrow{\ \mathrm{pr}_2\ } \mathbb{C}^{n+1}$$

where $\mathrm{pr}_1^{-1}(\ell) = \{u \in \mathbb{C}^{n+1} \mid u \in \ell\} \cong \mathbb{C}$ so $\tau \to \mathbb{C}P^n$ is a complex line bundle. While

$$\mathrm{pr}_2^{-1}(u) = \begin{cases} \ell_{0,u} := \mathbb{C}u & \text{if } u \neq 0 \\ \mathbb{C}P^n & \text{if } u = 0 \end{cases}$$

called the blow-up map.

Example 5.6. To make it possible to imagine, do everything over $\mathbb{R}$ and let $n = 1$. In this case,

$$\begin{array}{ccc} & \tau_{\mathbb{R}} = \big\{(\ell, u) \in \mathbb{R}P^1 \times \mathbb{R}^2 \mid u \in \ell\big\} \subseteq \mathbb{R}P^1 \times \mathbb{R}^2 & \\ \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} \\ \mathbb{R}P^1 & & \mathbb{R}^2 \end{array}$$

so the map $\mathrm{pr}_1 : \tau_{\mathbb{R}} \to \mathbb{R}P^1$ is a Möbius band and now pr_2 may be pictured a bit easier.

Example 5.7. Take $n = 1$ and the complex line bundle pr_1 over $\mathbb{C}P^1 \cong \mathbb{S}^2$. It can be given by another construction: Take $\mathbb{C}P^2$, fix the ideal line (together with its ideal point), it is a $\mathbb{C}P^1$ and fix a point $p \in \mathbb{C}P^2 \backslash \mathbb{C}P^1$. We define a map

$$\pi : \mathbb{C}P^2 \backslash \{p\} \to \mathbb{C}P^1$$
$$q \mapsto \mathbb{C}P^1 \cap L_q$$

where $L_q := \{\text{the line passing through } q\}$. We state that this is the same as the previous complex line bundle.

Remark 5.8. Now, if we would like to extend this map to p as well then we can realize that it is impossible since it should map to every point by continuity. So the solution is to "apply the blowing up" so we can replace p with a whole complex line and then the map becomes extendable. What does this "blowing up at p" means? Since locally every four dimensional complex manifold is $\mathbb{C}^2$, we can use the above construction as we fix p as the origin, we take a trivializing neighborhood and we work in that. The resulting manifold is called a Hirzebuch surface.

5.2 Chern classes

Definition 5.9. Suppose that $L \to X$ is a line bundle. The *total Chern class* is defined as

$$c(L) = 1 + c_1(L) \in H^*(X; \mathbb{Z})$$

where $1 \in H^0(X; \mathbb{Z})$ is the unit element of the ring and $c_1(L) \in H^2(X; \mathbb{Z})$ is the first Chern class of the line bundle L.

Suppose that $E \to X$ is a complex n-plane bundle (the fibers are $\mathbb{C}^n$) with the additional property that $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Then we define the Chern class as

$$H^*(X, \mathbb{Z}) \ni c(E) := \prod c(L_i) = \prod (1 + c_1(L_i)) = 1 + \sum_{i=1}^{n} c_1(L_i) + \sum_{i \neq j} c_1(L_i) \cup c_1(L_j) +$$

The k-th homogeneous components of this quantity are denoted by $c_k(E)$, so precisely,

$$c_k(E) := \sum_{\{i_1, \ldots, i_J\} \subseteq \{1, \ldots, n\}} c_1(L_{i_1}) \cup \cdots \cup c_1(L_{i_j})$$

called the k-th Chern class.

Remark 5.10. This assumption $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ would always hold if the image of the structure group in $GL_n(\mathbb{C})$ can be "reduced" to $GL_1(\mathbb{C}) \times \cdots \times GL_1(\mathbb{C})$, the subgroup of diagonal matrices. However, this does not always happen: Consider the tautological quaternionic line bundle over $\mathbb{H}P^1$:

$$\tau_{\mathbb{H}} = \left\{(\ell, u) \in \mathbb{H}P^1 \times \mathbb{H}^2 \mid u \in \ell\right\} \to \mathbb{H}P^1 \cong \mathbb{S}^4$$

with fibers $\mathbb{C}^2$. This bundle $\tau_{\mathbb{H}} \to \mathbb{S}^4$ is nontrivial since if it were then the unit vectors would give a map into $\mathbb{S}^4 \times \mathbb{S}^3 \to \mathbb{S}^4$. However, one can notice that $\tau_{\mathbb{H}} \cong \mathbb{S}^7$ and $\mathbb{S}^7 \ncong \mathbb{S}^4 \times \mathbb{S}^3$. But if $E = L_1 \oplus L_2$ then both L_1 and L_2 are trivial since L_i's are the line bundles characterized by their Chern classes in $H^2(\mathbb{S}^4, \mathbb{Z})$ which is the zero group.

Unfortunately, these already will not characterize all the higher dimensional vector bundles.

Definition 5.11. Suppose that $E \to X$ is a complex n-plane bundle (i.e. with fiber $\mathbb{C}^n$). A *splitting map* $f : Y \to X$ is a continuous map with two properties:

1. $f^*E = L_1 \oplus \cdots \oplus L_n$ (informally, we require Y to be smaller)
2. $f^* : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is a monomorphism and $\operatorname{Im} f^* \ni c(f^*E)$. (this means that Y is large enough)

Theorem 5.12. *For all $E \to X$ complex n-plane bundle there always exists a splitting map. (Provided that X is "reasonably nice".)*

Proof. (Idea of proof) Construct Y step-by-step as follows: First, consider the projectivization $\mathbb{P}(E) \to X$ of E. This can be defined as follows: The cocycle structure of the bundle $E \to X$ defines a principal bundle $P \to X$ with fibers $GL_n(\mathbb{C})$. We have seen in Subsection 4.1 that this principal vector bundle can induce $E = P \times_\rho \mathbb{C}^n$ where $\rho : GL_n(\mathbb{C}) \to \operatorname{Aut}(\mathbb{C}^n)$ is a representation. This can be factorized into a map $\rho_P : GL_n(\mathbb{C}) \to \mathbb{C}P^{n-1}$ so we can define $\mathbb{P}(E)$ as $P \times_{\rho_P} \mathbb{C}P^{n-1}$ which will be then a fiber bundle with fiber $\mathbb{C}P^{n-1}$ over X. Let's call its projection f. Now, we can pull back $E \to X$ along $\mathbb{P}(E) \to X$:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{P}(E) & \xrightarrow{\ f\ } & X \end{array}$$

By this we get that

Homework: (For 9th of March) $f^*E = L_1 \oplus E_1$ where L_1 is a line bundle and E_1 is already an $n-1$-plane bundle over $\mathbb{P}(E)$.

So we can repeat this process what eventually terminates. The harder part of the theorem is to check the properties of the obtained splitting map. ☐

Now, we give an axiomatic approach to the notion of total Chern class:

Properties: Suppose that $E \to X$ is a $\mathbb{C}^n$-bundle.

1. $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) \in H^*(X; \mathbb{Z})$ such that $c_i(E) \in H^{2i}(X; \mathbb{Z})$.
2. If $E \to X$ and $F \to X$ are isomorphic fiber bundles then $c(E) = c(F)$. More generally, if $F = f^*E$ for some $f : Y \to X$ then $\big(c(F) = \big)c(f^*E) = f^*c(E)$.
3. The total Chern class of $E \oplus F \to X$ is $c(E \oplus F) = c(E) \cup c(F) \in H^*(X; \mathbb{Z})$. (This is called the Whitney product formula.)
4. ("Normalization") For the tautological bundle $\tau_\infty \to \mathbb{C}P^\infty$ we have $c(\tau_\infty) = 1 + g \in H^*(\mathbb{C}P^\infty; \mathbb{Z})$ where g is the chosen generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

It is clear that the axioms determine the value on every bundle uniquely so we only need that such a thing exists. By the existence of the splitting map (Theorem 5.12) and Definition 5.9 it does.

Remark 5.13. *(without proof)* If we want to axiomatize the theory of Chern classes then the last property (taken as an axiom) can be replaced by requiring for $\tau_1 \to \mathbb{C}P^1$ we have $c(\tau_1) = 1+g \in H^*(\mathbb{C}P^1;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$.

5.3 Stiefel-Whitney classes

Axioms:

1. If $R \to X$ is a real n-plane bundle (i.e. with fiber $\mathbb{R}^n$) then $w(R) = 1 + c(R) + \cdots + w_n(R) \in H^*(X;\mathbb{Z}_2)$ such that $w_i(R) \in H^i(X;\mathbb{Z}_2)$.
2. If $f : Y \to X$ is a map and $R \to X$ is a real n-plane bundle then $w(f^*R) = f^*w(R)$.
3. $w(R_1 \oplus R_2) = w(R_1) \cup w(R_2) \in H^*(X;\mathbb{Z}_2)$.
4. $w(\tau_\infty) = 1 + x \in H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$ where x is the only one nontrivial element of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$.

Remark 5.14. The last axiom can be replaced by the assumption $w(\tau_1) = 1 + x \in H^*(\mathbb{R}P^1;\mathbb{Z}_2)$.

Theorem 5.15. (without proof) *There is a splitting principle for the Stiefel-Whitney classes.*

Definition 5.16. (*Grassmannians*) The classifying spaces ($\mathbb{S}^\infty \to \mathbb{R}P^\infty$ and $\mathbb{S}^\infty \to \mathbb{C}P^\infty$ for principal line bundles and $\tau_\mathbb{C}$, $\tau_\mathbb{R}$) can be described for n-plane bundles as well. Consider the space $\mathrm{Gr}_k(\mathbb{C}^n)$, the Grassmannian of k-planes in $\mathbb{C}^n$. This – as a manifold – can be constructed as $GL_n(\mathbb{C})/GL_k(\mathbb{C})\times GL_{n-k}(\mathbb{C})$. For $k = 1$ we get back $\mathbb{C}P^{n-1}$.

Similarly, we can define the tautological bundle for these too:

$$\tau_k(\mathbb{C}^n) = \big\{(p,x) \in \mathrm{Gr}_k(\mathbb{C}^n) \times \mathbb{C}^n \mid x \in p\big\} \subseteq \mathrm{Gr}_k(\mathbb{C}^n) \times \mathbb{C}^n$$

$$\mathrm{Gr}_k(\mathbb{C}^n) \xleftarrow{\ \mathrm{pr}_1\ } \tau_k(\mathbb{C}^n) \xrightarrow{\ \mathrm{pr}_2\ } \mathbb{C}^n$$

where $\mathrm{pr}_1 : \tau_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ is a k-plane bundle. We can also take its infinite version: $\tau_k(\mathbb{C}^\infty) = \{(p,x) \in \mathrm{Gr}_k(\mathbb{C}^n) \times \mathbb{C}^\infty \mid x \in p\} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$.

Theorem 5.17. *Any complex k-plane bundle $E \to X$ can be pulled back from $\tau_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ that is, there exist $f : X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ such that $E = f^*\tau_k(\mathbb{C}^\infty)$.*

Theorem 5.18. $H^*\big(\mathrm{Gr}_k(\mathbb{C}^\infty);\mathbb{Z}\big) \cong \mathbb{Z}[c_1,\dots,c_k]$ *where* $c_i \in H^{2i}(\mathrm{Gr}_k(\mathbb{C}^\infty);\mathbb{Z})$.

Proposition 5.19. *For the tangent bundle $T\mathbb{R}P^n \to \mathbb{R}P^n$ its Stiefel-Whitney class is $(1+x)^{n+1} \in H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$.*

One should note that $x^{n+1} = 0$ by the structure of $H^*(\mathbb{R}P^n;\mathbb{Z}_2)$. About the other homogeneous terms, it is harder to decide whether they are zero or not since we are in characteristic two.

Corollary 5.20. $W(T\mathbb{R}P^n) = 1$ *if and only if* $n + 1 = 2^k$.

Proof. Clearly for $n + 1 = 2^k$ it works since squaring in characteristic two is additive.

Homework: (For 9th of March) Prove that $W(T\mathbb{R}P^n) = 1$ implies $n + 1 = 2^k$.

□

Now, to motivate Proposition 5.19, we show an application of it:

Theorem 5.21. (Whitney embedding theorem) *If M^n is an n-dimensional manifold then $M^n \hookrightarrow \mathbb{R}^N$.*

Question: What is the optimal N? Sharpening of this theorem gives $M^n \hookrightarrow \mathbb{R}^{2n-1}$.

Theorem 5.22. *If $\mathbb{R}P^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$ then $k \geq 2^r - 1$.*

Proof. With $n = 2^r$ we have $W(T\mathbb{R}P^{2^r}) = (1+x)^{2^r+1} = (1+x)(1+x)^{2^r} = (1+x)(1+x^{2^r}) = 1 + x + x^{2^r}$ since $x^{2^r+1} = 0$. Why is it useful? We will see.

Suppose that $\mathbb{R}P^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$. We know that $T\mathbb{R}^{2^r+k} \cong \mathbb{R}^{2^r+k} \times \mathbb{R}^{2^r+k}$. So we can view

$$T\mathbb{R}P^{2^r} \subseteq T\mathbb{R}^{2^r+k} \cong \mathbb{R}^{2^r+k} \times \mathbb{R}^{2^r+k}$$

In fact, we get a decomposition $T\mathbb{R}^{2^r+k}|_{\mathbb{R}P^{2^r}} = T\mathbb{R}P^{2^r} \oplus \nu$ using a tangential and normal decomposition of the bundle (It can be done locally). We also know that this bundle is trivial since it is the restriction of a trivial bundle. Now, let's take its Stiefel-Whitney class:

$$1 = W\big(T\mathbb{R}^{2^r+k}|_{\mathbb{R}P^{2^r}}\big) = W(T\mathbb{R}P^{2^r} \oplus \nu) = W(T\mathbb{R}P^{2^r}) \cup W(\nu) \in H^*(\mathbb{R}P^{2^r}; \mathbb{Z}_2)$$

yielding the equation $1 = (1 + x + x^{2^r})W(\nu) \in \mathbb{Z}[x]/(x^{2^r+1})$. This can be solved, its unique (!) solution is $W(\nu) = 1 + x + x^2 + \cdots + x^{2^r-1}$. (It is straightforward to compute.)

This means that $k \geq 2^r - 1$ since $W(\nu)$ must have nontrivial component in degree $2^r - 1$. $\square$

Proof. of Proposition 5.19:

Step1: Consider the tautological bundle $\tau \to \mathbb{R}P^n$. This is the subbundle of the trivial $\mathbb{R}P^n \times \mathbb{R}^{n+1} \to \mathbb{R}P^n$ bundle so we can take an (orthogonal) complement $\tau^\perp$ of τ such that $\tau \oplus \tau^\perp = \mathbb{R}P^n \times \mathbb{R}$. One can show that

$$T\mathbb{R}P^n \cong \mathrm{Hom}(\tau, \tau^\perp)$$

as vector bundles.

Step2: Consider $\mathrm{Hom}(\tau, \tau)$ which is a trivial 1 dimensional bundle since it has a nonzero section (by taking id in every fiber). So we can compute the "value" of the following bundle (where ε is the trivial line bundle):

$$T\mathbb{R}P^n \oplus \varepsilon = \mathrm{Hom}(\tau, \tau^\perp) \oplus \mathrm{Hom}(\tau, \tau) = \mathrm{Hom}(\tau, \tau^\perp \oplus \tau) = \mathrm{Hom}(\tau, \varepsilon^{n+1}) = \oplus_{i=1}^{n+1} \tau^*$$

what has Stiefel-Whitney class $(1+x)^{n+1}$. This means that $W(T\mathbb{R}P^n \oplus \varepsilon) = W(\oplus_{i=1}^{n+1} \tau^*)$ what implies $W(T\mathbb{R}P^n) = W(\tau)^{n+1} = (1+x)^{n+1}$.

$\square$

n-th Lecture, 23th of March

5.4 Application of Stiefel-Whitney classes

Suppose that M is a closed n-dimensional smooth manifold such that $H_n(M, \mathbb{Z}_2) \cong \langle \mu_M \rangle$. Recall that

$$w_i(M) := w_i(TM) \in H^i(M, \mathbb{Z}_2) \qquad i = 0, 1, \ldots, n$$

Suppose, moreover, that $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $\sum_{k=1}^n k \cdot i_k = n$. Then

$$w_I(M) = w_{i_1}(M) \cup \cdots \cup w_{i_n}(M) \in H^n(M; \mathbb{Z}_2)$$

i.e. we defined all the possible cup products of the Stiefel-Whitney classes arriving in $H^n(M; \mathbb{Z}_2)$.

Definition 5.23. Define the *Stiefel-Whitney number* $w_I\big([M]\big) \in \mathbb{Z}_2$ as $\langle w_I(M), \mu_M \rangle \in \mathbb{Z}_2$.

Example 5.24. $W(\mathbb{R}P^n) = \sum_{i=0}^n w_i(T\mathbb{R}P^n) = (1+x)^{n+1}$ where $x \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the generator of the group (in fact it is the ring-generator of the cohomology ring $H^*(\mathbb{R}P^n) = \mathbb{Z}_2[x]/(x^{n+1})$).

If n is even then $w_n(\mathbb{R}P^n) = (n+1)x^n = x^n$, i.e. $\langle w_n(\mathbb{R}P^n), \mu_{\mathbb{R}P^n}\rangle = 1$. By the above notation, this is $w_{(0,0,\dots,1)}[\mathbb{R}P^n] = 1$. Moreover,

$$w_1^n\big(\mathbb{R}P^n\big) = \big((n+1)x\big)^n = x^n$$

so $\langle w^n(\mathbb{R}P^n), \mu_{\mathbb{R}P^n}\rangle = 1$ which means $w_{(n,0,\dots,0)}[\mathbb{R}P^n] = 1$. In fact, if $n = 2^r$ then these are the only non-zero Stiefel-Whitney classes.

Proposition 5.25. *If n is odd then $w_I[\mathbb{R}P^n] = 0$.*

Proof. Write $n = 2k - 1$ then

$$W(\mathbb{R}P^n) = (1+x)^{n+1} = (1+x)^{2k} = (1+x^2)^k$$

hence all $w_i(\mathbb{R}P^n) = 0$ for odd i. However, $w_I[\mathbb{R}P^n]$ always contain an odd i since n is odd (even numbers don't sum to odd). Therefore, the cup product of them is always zero. ☐

Proposition 5.26. *Suppose that $M^n = \partial B^{n+1}$ where B^{n+1} is a smooth and compact $n+1$-dimensional manifold with boundary. Then $w_I[M] = 0$ for all I.*

Proof. Consider the fundamental class $\mu_M \in H_n(M; \mathbb{Z}_2)$ and $\mu_B \in H_{n+1}(B; \mathbb{Z}_2)$. In the long exact sequence of (B, M) we get a boundary morphism

$$\partial : H_{n+1}(B, M; \mathbb{Z}_2) \to H_n(M; \mathbb{Z}_2)$$

such that $\partial \mu_B = \mu_M$.

Note that $TB|_M = TM \oplus \varepsilon$ where the decomposition is not canonical. Consider the Stiefel-Whitney classes $w_i(B) := w_i(TB)$ and restrict them to M. (Precisely, this means applying $j^* : H^*(B) \to H^*(M)$ the induced map of the embedding $j : M \to B$.) Then we get

$$w_i(B)|_M = w_i(TB)|_M = w_i(TB|_M) = w_i(TM \oplus \varepsilon) = w_i(TM) = w_i(M)$$

Now, let's compute the Stiefel-Whitney number:

$$w_I[M] = \langle w_I(M), \mu_M\rangle = \langle w_I(M), \partial\mu_B\rangle = \langle w_I(B)|_M\big), \partial\mu_B\rangle = \langle \delta w_I(B), \mu_B\rangle = \langle 0, \mu_B\rangle$$

where the middle equality is valid by the definition of δ (i.e. it is composing by ∂ from the inside) and the last is true by $w_I(B) \in H^n(B; \mathbb{Z}_2)$. ☐

Theorem 5.27. (Thom) *The converse also holds: if M is a smooth and compact n-dimensional manifold such that $w_I[M] = 0$ for all I then $M = \partial B^{n+1}$ for some smooth compact manifold with boundary B.*

Homework: $\mathbb{R}P^1 = \partial ?$, $\mathbb{R}P^3 = \partial ?$,

Remark 5.28. Actually $\mathbb{R}P^{2k-1}$ is a boundary of something but that is more involved.

5.5 Cobordism groups

Definition 5.29. Consider smooth closed (compact without boundary) n-manifolds up to diffeomorphism M_1^n and M_2^n are *equivalent* or *cobordant* if there is W^{n+1} such that $\partial W = M_1 \cup M_2$.

Proposition 5.30. *The set of equivalence classes is a group (denoted by $\mathfrak{N}_n$) with respect to disjoint union as the operation and empty set as the identity.*

Remark 5.31. In fact, it is an abelian group, as one can easily see, which is also 2-torsion. This means that it is a vector space over $\mathbb{Z}_2$ and we can compute its dimension using the Stiefel-Whitney numbers.

Variations:

1. $\mathfrak{N}_* := \bigoplus_{n\geq 0} \mathfrak{N}_n$ can be equipped with a ring structure using $(M^n, N^m) \mapsto (M \times N)^{n+m}$.
2. If we also assume the manifolds to be oriented then we need another notion of equivalence: $M_1 \sim M_2$ if and only if there exists a smooth, compact and oriented $(n+1)$-manifold W^{n+1} such that $\partial W^{n+1} = M_2 \cup (-M_1)$ where by $-M_1$ we mean the same smooth manifold by reversed orientation (i.e. the negative of the previously fixed generator of the top homology). Here, we use the convention "outward normal first" (opposed to inward normal first, outward normal last, or inward normal last) when the boundary inherits an orientation from the bigger one. This leads to the oriented cobordism group Ω_n. This is already not necessarily 2-torsion.

   Example 5.32. $\Omega_0 = \mathbb{Z}$, $\Omega_1 = \Omega_2 = 0$ since all 1- and 2-dimensional oriented manifold is a boundary. In fact, $\Omega_3 = 0$ too. However,

   $$\Omega_4 \cong \mathbb{Z} \cong \langle \mathbb{C}P^2 \rangle$$

   $$\Omega_5 \cong \mathbb{Z}_2 \qquad \Omega_6 \cong \Omega_7 \cong 0$$

   $$\Omega_8 \cong \mathbb{Z} \oplus \mathbb{Z} \cong \langle \mathbb{C}P^2 \times \mathbb{C}P^2, \mathbb{C}P^4 \rangle$$

   $$\Omega_9 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \qquad \Omega_{10} = \Omega_{11} \cong \mathbb{Z}_2$$

   Moreover, there is a ring structure on the direct sum Ω_* called oriented cobordism ring, defined analogously to $\mathfrak{N}_*$.

   Theorem 5.33. (Thom) $\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[x_1, x_2, \ldots, x_n, \ldots]$ *where* $x_i = [\mathbb{C}P^{2i}]$ *with degree* $4i$*.*
3. Fix a topological space X. Consider the set of pairs (f, M) such that $f : M \to X$ is continuous where M is a smooth closed manifold. Define $(f_1, M_1) \sim (f_2, M_2)$ named cobordant if there exists an (F, W) such that W has dimension $n+1$ and $\partial W = M_2 \cup (-M_1)$ and $F : W \to X$ is a continuous maps such that $F|_{M_1} = f_1$ and $F|_{M_2} = f_2$. This concept gives the bordism groups $\Omega_n(X)$ for X.

5.6 Explicit computation for Chern classes

Let $S_d = \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid \sum_{i=0}^{3} z_i^d = 0\right\}$ which is a complex surface (i.e. real 4-manifold). By the implicit function theorem, it is indeed a smooth complex compact closed submanifold.

Facts: First, note that $\pi_1(S_d) = 1$. Moreover, we know that any two degree d polynomials satisfying the implicit function theorem provide diffeomorphic submanifolds. This assumption means that their Jacobian has full rank, i.e. it is a submersion. Fortunately, the subset of polynomials failing to satisfy this assumption is a complex codimension 1 subspace hence real codimension 2 meaning that it is so small that we can always avoid it with a continuous path between the two given polynomials.

Goal: to compute $c_1(TS_d)$ and $c_2(TS_d)$.

Recall that $c(T\mathbb{C}P^3) = (1+g)^4$ where g is the canonical generator of $H^2(\mathbb{C}P^3; \mathbb{Z}) \cong \mathbb{Z}_2$ the one giving $\langle g, [\mathbb{C}P^3] \rangle = 1$.

Proposition 5.34. *Let us denote* i^*g *by* $x \in H^2(S_d; \mathbb{Z})$ *where* $i : S_d \hookrightarrow \mathbb{C}P^3$ *is the natural embedding. Then*

1. $c_1(S_d) = (4-d)x \in H^2(S_d; \mathbb{Z})$
2. $c_2(S_d) = (d^2 - 4d + 6)x^2 \in H^4(S_d; \mathbb{Z})$
3. $\langle x^2, [S_d] \rangle = d$.

Hence, $c_2[S_d] = (d^2 - 4d + 6)d$ *and* $c_1^2[S_d] = (4-d)^2 d$.

Proof. We know that $c(T\mathbb{C}P^3) = (1+g)^4 = 1 + 4g + 6g + 4g^3$. Moreover,

$$T\mathbb{C}P^3|_{S_d} = TS_d \oplus \nu S_d$$

where νS_d is the normal bundle of S_d. This means that

$$c(T\mathbb{C}P^3|_{S_d}) = c(i^*T\mathbb{C}P^3) = i^*c(T\mathbb{C}P^3) = i^*(1+g)^4 =$$

but i^* is a ring homomorphism hence

$$= (1+x)^4 = 1 + 4x + 6x^2 + 0 \in \mathbb{Z}_2[x]/(x^3)$$

On the other hand,

$$c(T\mathbb{C}P^3|_{S_d}) = c(TS_d \oplus \nu S_d) = c(TS_d) \cup c(\nu S_d) = \big(1 + c_1(S_d) + c_2(S_d)\big)\big(1 + c_1(\nu S_d)\big)$$

So we can conclude that

$$\big(1 + c_1(S_d) + c_2(S_d)\big) = (1 + 4x + 6x^2)\big(1 + c_1(\nu S_d)\big)^{-1}$$

where we want the inverse in the cohomology ring. Since it has nonzero constant term, it has an inverse which is $1 - c_1(\nu S_d) + c_1^2(\nu S_d)$ what can be checked by direct computation.

Claim 5.35. $c_1(\nu S_d) = dx$

Proof. (sketch) Consider S_d' given by a "nearby" degree d homogeneous polynomial. This gives a section of the normal bundle $\nu S_d \to S_d$. Consider $V = S_d \cap S_d'$. By general nonsense on Poincaré dual, it is possible to know that $PD\big(c_1(\nu S_d)\big) = [V]$. So $[S_d] = d[S_1] \in H_4(\mathbb{C}P^3; \mathbb{Z}) \cong \mathbb{Z}$. Now, consider the dual of the intersection with a complex line $\mathbb{C}P^1 \subseteq \mathbb{C}P^3$:

$$PD[S_1 \cap S_d] = i^*g = x$$

hence

$$PD[V] = PD[S_d \cap S_d'] = dx$$

Besides, we can express

$$\langle x^2, [S_d]\rangle = \langle (i^*g)^2, [S_d]\rangle = \langle g^2, i_*[S_d]\rangle \overset{\text{black magic}}{=} \langle g^2 \cup PD\big(i_*[S_d]\big), [\mathbb{C}P^3]\rangle = \langle g^2 \cup dg, [\mathbb{C}P^3]\rangle = d$$

giving the claim. ☐

By the claim, we get

$$\big(1 + c_1(S_d) + c_2(S_d)\big) = (1 + 4x + 6x^2)(1 - dx + d^2x^2) = 1 + (4-d)x + (d^2 - 4d + 6)x^2$$

so we proved the proposition. ☐

Theorem 5.36. (Friedman) *The homeomorphism type of a smooth simply connected* 4*-manifold is determined by* $H^*(X;\mathbb{Z})$.

Remark 5.37. Since it is simply connected and oriented, we get $H^1 = H^3 = 0$ so what is left is H^2 and the ring structure. In fact, $H^*(X;\mathbb{Z})$ can be – in principle – characterized by two integers: $\chi = 2 + b_2$ where b_2 is the second Betti number and σ the signature we will introduce later.

Facts: $c_2[S_d] = \chi(S_d)$ and $c_1^2[S_d] = 3\sigma(S_d) + 2\chi(S_d)$.

The above procedure on computing the Chern numbers can be generalized to the complex surfaces

$$S_{d_1,\dots,d_{n-2}} = \{z \in \mathbb{C}P^n \mid p_i(z) = 0,\ i = 1, \dots, n-2\}$$

where p_i is a degree d_i homogeneous polynomial.

Corollary 5.38. $(3,3,6,7,7,10)$ *and* $(2,2,3,3,3,3,3,3,5,9)$ *are homeomorphic* 4*-manifolds by Freeman's Theorem.*

However, the Chern classes have different divisibility properties hence they are not diffeomorphic.

5.7 Application of Chern classes

Chern classes are defined only for manifolds that have a complex structure on their fibers. Therefore, if we want to apply our knowledge about Chern classes on a real vector bundle, it is reasonable to complexify it. In details, this means that for a real vector bundle $F \to X$ with fibers $\mathbb{R}^n$ we can take $F \otimes \mathbb{C}$ in every trivializing chart, so $F \otimes \mathbb{C}$ becomes a complex n-plane bundle.

On the language of cocycle structures it means that the elements $g_{\alpha\beta} \in GL(\mathbb{R})$ can be viewed as elements of $GL(\mathbb{C})$ using the standard embedding $GL(\mathbb{R}) \hookrightarrow GL(\mathbb{C})$.

Among complex vector bundles, we have a new operation: taking conjugation $E \mapsto \overline{E}$. In our special complexified case the conjugate of the vector bundle is the vector bundle itself. Therefore, the Chern classes will have more rigid form:

$$1 + c_1(E) + c_2(E) + \cdots =: c(E) = c(\overline{E})$$

where the right hand side can be computed by the following Lemma:

Lemma 5.39. $c(\overline{E}) = 1 - c_1(E) + c_2(E) - \dots$

Corollary 5.40. $2c_i(E) = 0$ *for all odd* $i \in \mathbb{N}$.

Definition 5.41. Let us define the *Pontrjagin class* of the bundle $F \to X$ as

$$p_i(F) = (-1)^i c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z})$$

Moreover, we can define the *total Pontrjagin class* as

$$p(F) := 1 + p_1(F) + p_2(F) + \cdots + p_{\left[\frac{n}{2}\right]}(F) \in H^*(X; \mathbb{Z})$$

Facts:

1. $p(F \otimes \underline{\mathbb{R}}^n) = p(F)$
2. $2\big(p(E \oplus F) - p(E) \cup p(F)\big) = 0$
3. If F is already a complex vector bundle then

$$\big(1 - c_1(F) + c_2(F) - c_3(F) + \dots\big) \cdot \big(1 + c_1(F) + c_2(F) + \dots\big) =$$

$$= 1 - p_1(E) + p_2(E) - p_3(E)$$

4. $p(\mathbb{C}P^1) = 1$, $p(\mathbb{C}P^2) = 1 + 3g^2$, $p(\mathbb{C}P^3) = 1 + 4g^2$, $p(\mathbb{C}P^4) = 1 + 5g^2 + 10g^4$.

Definition 5.42. Analogously to the Stiefel-Whitney case, we can define the Pontrjagin numbers in a similar fashion as we have done at the numbers $w_I[M]$. Namely, by the formula

$$p_I[M] := \langle p_I(TM), \mu_M \rangle$$

where μ_M is the fundamental class of M and $I = (i_1, i_2, \dots)$ is a sequence where $\sum_k 4k \cdot i_k = \dim M$. In particular, we must assume that $\dim M$ is divisible by four, otherwise there is no Pontrjagin number on the manifold.

Definition 5.43. Suppose that M^{4n} is a closed oriented smooth manifold. Then we can define the bilinear form

$$Q_M : H^{2n}(M; \mathbb{R}) \times H^{2n}(M; \mathbb{R}) \longrightarrow \mathbb{R}$$

$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, \mu_M \rangle$$

This becomes a bilinear form since $\cup$ is bilinear. It is also symmetric, since $\cup$ is graded-commutative and now the dimension is even.

Remark 5.44. Note that this bilinear form is never degenerate by the Poincaré duality.

Symmetric bilinear forms over $\mathbb{R}$ are characterized by the signature (see Sylvester's theorem). Therefore, in this non-degenerate case, it is enough to give the rank $b_{2n} := \mathrm{rk}Q_M$ and say b_{2n}^+ the number of $+1$'s in the bilinear form. Analogously, we can define b_{2n}^- the number of -1's and then $\sigma = b_{2n}^+ - b_{2n}^-$.

Remark 5.45. In the case of $\mathbb{Z}$ the classification of "bilinear forms" are more advanced. It is "because" in that case we do not have the nice realization of cohomology as differential forms (see de Rham cohomology).

Facts:

1. $\sigma(M_1 \cup M_2) = \sigma(M_1) + \sigma(M_2)$
2. $\sigma(M_1 \times M_2) = \sigma(M_1) + \sigma(M_2)$
3. $M = \partial W$ then $\sigma(M) = 0$.

Corollary 5.46. $\sigma : \Omega_* \to \mathbb{Z}$ *is well-defined and it is an algebra homomorphism.*

Theorem 5.47. (Hirzebuch signature theorem) $\sigma(M)$ *can be determined by the Pontrjagin numbers.*

Example 5.48.
For $n = 4$: $\sigma(M) = \frac{1}{3}p_1[M] \in \mathbb{Z}$
For $n = 8$: $\sigma(M) = \frac{1}{45} \cdot \big(7p_2[M] - p_1^2[M]\big) \in \mathbb{Z}$
For $n = 12$: $\sigma(M) = \frac{1}{945}\Big(62p_3[M] + 3p_1p_2[M] + 8p_1^3[M]\Big)$

Let $A = (A^0, A^1, A^2, \dots)$ be a sequence of Abelian groups (we think about them as $H^i(M)$ or Ω_i) where $A = \prod_{i=0}^\infty A^i$ is also a ring. Let

$$\underline{K} = \{K_1(x_1), K_2(x_1, x_2), \dots\}$$

be a sequence of polynomials on $\prod_{i=0}^n A^i$. We call it a multiplicative sequence, if

- $K_n(x_1, \dots, x_n)$ is homogeneous of degree n
- Let $a \in A$ then we define $K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \dots$
- $K(a \cdot b) = K(a) \cdot K(b)$ where the latter is the Cauchy product of A.

Proposition 5.49. *A formal power series*

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \dots + \lambda_n t^n + \dots$$

uniquely determines a multiplicative sequence by the rule

$$K(1 + t) = f(t)$$

where t is a formal indeterminate.

Example 5.50. $f(t) = \frac{\sqrt{t}}{\mathrm{th}(\sqrt{t})} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \dots + (-1)^{k-1}2^{2^k}B_k\frac{t^k}{(2k)!}$ where B_k is the k-th Bernoulli number $B_k \in \mathbb{Q}$.

Theorem 5.51. (Hirzebuch signature theorem, take 2) *The multiplicative sequence corresponding to the* $f(t)$ *above gives* σ.

Idea: $f(t)$ gives us a multiplicative sequence L, and this L gives an $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$ ring-homomorphism. Similarly, $\sigma : \Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$ is a ring homomorphism. Hence, it is enough to see that they agree on the generators, namely, we have to compute that

$$\sigma(T\mathbb{C}P^{2k}) = 1$$

for all $k \in \mathbb{N}$, and similarly, the same should be computed for L.

Remark 5.52. By this result, one can obtain an exotic sphere, in fact the first exotic sphere was obtained this way by Milnor in 1956.

Consider S_d as in Subsection 5.6. Then we may know (without proof) that $c_2[S_d] = \chi(S_d)$. Moreover, we have seen that $p_1 = c_1^2 - 2c_2$ so it can be determined. By the signature theorem, we know that

$$\sigma = \frac{1}{3}p_1$$

Moreover, $\pi_1(S_d)$ is known to be trivial. The statement is that this information is enough to determine $H^*(S_d;\mathbb{Z})$. Indeed,

$$c_2[S_d] = \chi(S_d) = b_0 + b_1 + b_2 + b_3 + b_4 = 2 + b_2$$

hence $c_2[S_d]$ determines b_2. Therefore, we know all the groups appearing in $H^*(S_d;\mathbb{Z})$ and we just need to determine the ring structure. This is completely determined by the bilinear form $H^2 \times H^2 \to \mathbb{Z}$ since all other products are trivial. In our case, the bilinear form is known to be not definite (easy to check) and non-degenerate. Such forms are classified by the following theorem:

Theorem 5.53. *If Q is a symmetric bilinear form over $\mathbb{Z}$ that is not definite and non-degenerate. Then either*

$$Q = n \cdot \langle 1 \rangle \oplus m\langle -1 \rangle \qquad \text{or} \qquad Q = kE_8 \oplus lH$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is a unique 8-dimensional symmetric bilinear form.

Since we know the signature, there are at most 2 choice for the bilinear form. The difference between the two choices is whether c_1 is even or odd. Therefore, Q is completely determined using the deduced equations i.e. the relations between p_i, c_i, σ and $\chi(S_d)$.

Remark 5.54. The gain in determining the cohomology ring of this manifold is more visible by the theorem stating that for simply connected 4-manifolds, the cohomology ring is a completely characterizing invariant for topological (!) manifolds. However, we can construct another 4-manifold with the same cohomology ring (i.e. it is homeomorphic) which is not diffeomorphic, namely

$$T_d := \bigoplus_{i=1}^{n} \mathbb{C}P^2 \oplus \bigoplus_{i=1}^{m} \overline{\mathbb{C}P^2}$$

Definition 5.55. A topological manifold M^n admits a unique differentiable structure if any two differentiable structure on it is diffeomorphic.

For $n \leq 3$ it is true for every manifold. For $M^n = \mathbb{R}^4$ it is already not. To prove this, we list three theorems:

- If M^4 is a non-compact topological 4-manifold then it admits a smooth structure. (without proof)
- Moreover, there exists a $K \subseteq \mathbb{S}^3$ knot such that in D^4 it bounds a topological D^2 but not a smooth D^2. To prove the first, we only have to check that the Alexander polynomial of K is trivial.
- For the second statement, i.e. it is not smooth: there exists an invariant $\tau(K) \in \mathbb{Z}$ what can prove that something does not bound a smooth D^2.

By these, one can construct non-diffeomorphic differentiable structures on $\mathbb{R}^4$.