Language of Schemes

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Remark. This is the live-texed notes of Language of Schemes course held by Tamás Szamuely in the winter of 2016. Any mistakes and typos are my own.

FIRST LECTURE, 14TH OF JANUARY

Literature:

- Schemes: The Beginnings (notes on the website)
- Cohomology of coherent schemes (notes on the website)
- Hartshorne: Algebraic Geometry, Chapter II-III.
- Mumford: The Red Book of Varieties and Schemes, and Algebraic Geometry II
- Grothendieck: EGA
- Görtz - Wedhorn: Algebraic Geometry I (and II when it appears)
- Liu: Algebraic Geometry and Arithmetic Curves
- de Jong: Stacks Project (as a reference book)

Outline:

1. Definitions
2. Cohomology of coherent schemes
3. Chow's Theorem
4. Serre's duality
5. Riemann-Roch theorem
6. Jacobi variety of curves

1 Definitions of Schemes

The definition of scheme arises as a generalization of an algebraic variety. It is abstracted as it will not have a fixed embedding space in contrast to the algebraic varieties. The given data for it is a topological space and the rings of functions on each open subset. First, we will define the affine varieties and then a general scheme will be defined as something that is patched together from affine pieces. For this plan, we will need the machinery of sheaves.

1.1 Sheaves

Definition 1.1. Let X be a topological space. A *presheaf* of abelian groups on X is a function $\mathcal{F}$ that associates an abelian group $\mathcal{F}(U)$ to each open subset $U \subseteq X$ in a way that for every open inclusion $V \subseteq U \subseteq X$ then we have a natural associated $\mathcal{F}(U) \to \mathcal{F}(V)$ is a homomorphism and this assignment keeps units and compositions. We also assume that $\mathcal{F}(\emptyset) = 0$.

In the terminology of category theory, we can equivalently define $\mathcal{F}$ as a contravariant functor to the category of abelian groups (with group homomorphisms) from the category that has the open subsets of X as objects and only the inclusions as morphisms. The latter category is now denoted by $\mathcal{C}_X$.

Example 1.2. A basic example is when $\mathcal{F}(U)$ is the set of continuous functions to $\mathbb{R}$ for each open set $U \subseteq X$, and the morphisms are the restrictions. Many other types of functions also give us a sheaf, together with the restrictions. Hence, the morphisms $\mathcal{F}(V \hookrightarrow U)$ will be denoted as *restrictions*, and the elements of $\mathcal{F}(U)$ are called *sections*.

Definition 1.3. A *sheaf* on X is a presheaf $\mathcal{F}$ satisfying the following two axioms:

1. Given an open covering $\{U_i\}_{i\in I}$ of a set U and a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for all i then $s = 0$.

2. Given an open covering $\{U_i\}_{i\in I}$ of a set U and for all i an $s_i \in \mathcal{F}(U)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ then there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s$.

Note that the first assumption gives uniquenesss in the second.

Remark 1.4. The axioms of a sheaf are equivalent to the following: for any open covering $\{U_i\}_{i\in I}$ of a set U the following sequence is exact:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\text{restr.}} \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

where the last two arrows are the products of restrictions $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$ and $\mathcal{F}(U_i) \to \mathcal{F}(U_j \cap U_i)$, respectively.

Example 1.5.

1. A counterexample, the constant *presheaf* (which is not always a sheaf): Fix an abelian group A and define $\mathcal{F}_A(U) := A$ for all nonempty open subsets U and $\mathcal{F}(U \hookrightarrow V) = \mathrm{id}_A$ for all $U \subseteq V$. If X is not irreducible, i.e. we have two nonempty disjoint open sets U and V then $1 \in \mathcal{F}_A(U)$ and $0 \in \mathcal{F}_A(U)$ are compatible that have no common extension to $U \cup V$.

2. A correction to it, the *constant sheaf*: fix an abelian group A and define $\mathcal{F}(U) = A^{\oplus n}$ where n is the number of connected components of the open set U.

3. The *skyscraper sheaf*: Fix an abelian group A and $p \in X$. Define $\mathcal{F}^p(U)$ as A if $p \in U$ and 0 if $p \in U$.

Remark 1.6. Similarly, one can define sheaves of rings, of modules (over a fixed ring), of sets. (In the last case, one has to be a bit more careful with the first axiom of sheaf, as we have no subtraction or zero.)

Definition 1.7. A *morphism of presheaves* on the same space X is a morphism or natural transformation of (contravariant) functors, i.e. for all open set U we take a morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for all inclusion $V \hookrightarrow U$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

A morphism of sheaves is just a morphism of their underlying presheaves.

Given a morphism of presheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$, $\mathrm{Ker}(\varphi)$ is the presheaf given by the (pointwise) kernels $\mathrm{Ker}(\varphi_U)$. Similarly, $\mathrm{Im}(\varphi)$ can be defined by $\mathrm{Im}(\varphi_U)$ for all U.

Remark 1.8. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves then $\mathrm{Ker}(\varphi)$ is also a sheaf. However, $\mathrm{Im}(\varphi)$ is not necessarily, the second axiom of sheaves will sometimes fail.

Proposition 1.9. *Given a presheaf $\mathcal{F}$ on X, there exists a sheaf $\mathcal{F}^\sharp$ on X and a morphism $\mathcal{F} \to \mathcal{F}^\sharp$ such that the following diagram can always be completed in a unique way:*

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^\sharp \\ & \overset{\rho}{\searrow} & \downarrow \tilde{\rho} \\ & & \mathcal{G} \end{array}$$

In other words, it is a left adjoint to the forgetful functor Sheaves $\to$ Presheaves*.*

Definition 1.10. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves then the image sheaf $\mathrm{Im}(\varphi)$ is defined as $\left(\mathrm{Im}_{\mathrm{presh}}(\varphi)\right)^\sharp$.

Definition 1.11. A *stalk* of a presheaf is defined as follows: Let $\mathcal{F}$ be a presheaf on a topological space X, a point $p \in X$. Then the stalk of $\mathcal{F}$ at p is

$$\mathcal{F}_p := \left\{(U, s) \mid U \subseteq X \text{ open } p \in U,\ s \in \mathcal{F}(U)\right\} / \sim$$

where $(U, s) \sim (U', s')$ if and only if there exists a $p \in V \subseteq U \cap U'$ open such that $s|_V = s'|_V$. In categorical terms, it is the same as $\mathcal{F}_p = \lim_{\to U \ni p} \mathcal{F}(U)$ where the direct limit uses the directed system of open sets with the reverse inclusion relation.

Proof. The construction of $\mathcal{F}^\sharp$ is the following: For an open subset $U \hookrightarrow X$ we take $\mathcal{F}^\sharp$ the compatible germs, i.e. the maps

$$U \to \cup^*_{p \in U} \mathcal{F}_p \qquad p \mapsto s_p \in \mathcal{F}_p$$

such that there exists an open covering $\left\{U_i\right\}_{i \in I}$ of U such that there exist $s_i \in \mathcal{F}(U_i)$ such that $\forall p \in U_i$, s_p is the image of s_i in $\mathcal{F}_p$. It is automatic that $\mathcal{F}^\sharp$ is a sheaf, "as we patch together rings of functions".

Moreover, note that $\left(\mathcal{F}^\sharp\right)_p = \mathcal{F}_p$ and it has the stated universal property: for any $s \in \mathcal{F}^\sharp(U)$ take an open covering $\left\{U_i\right\}_{i \in I}$ of U and $s_i \in \mathcal{F}(U_i)$, then we can associate $\rho(s_i) \in \mathcal{G}(U_i)$ to it. These are compatible by definition hence there exists a $\tilde{\rho}(s) \in \mathcal{G}(U)$. □

Example 1.12. What are the stalks of the skyscraper sheaf? Clearly, $(\mathcal{F}^p)_p = A$ but we also have

$$(\mathcal{F}^p)_q = \begin{cases} A & \text{if } q \in \overline{\{p\}} \\ 0 & \text{otherwise} \end{cases}$$

for $q \in X$ as one can check.

Having defined $\mathrm{Im}(\varphi)$ for a morphism of sheaves, we may define a sequence of morphisms

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow \dots$$

to be *exact*, if $\mathrm{Ker}(\varphi_i) = \mathrm{Im}(\varphi_{i-1})$ for all i.

Exercise 1.13. (Homework sheet I) A sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is exact, if and only if for all $p \in X$, the sequence

$$0 \longrightarrow \mathcal{F}_p \longrightarrow \mathcal{G}_p \longrightarrow \mathcal{H}_p \longrightarrow 0$$

with the induced morphisms is exact.

1.2 Affine schemes

An easy motivation of the notion of schemes, especially why we need arbitrary rings in the definition and not just finitely generated reduced $\Bbbk$-algebras, is the example of the parabola. The coordinate ring of the parabola $y^2 = t$ is $\Bbbk[y,t]/(y^2 - t)$. When one wants to investigate the fibers of the projection to the t-axis algebraically then one can substitute $a = t$ where $a \in \Bbbk$. Then we get $\Bbbk[y^2]/(y^2 - a) \cong \Bbbk \oplus \Bbbk$ if $a \neq 0$ but we get $\Bbbk[y]/(y^2)$ for $a = 0$. This ring is no more an integral domain but rather its nilpotent element stands for the clear geometric fact that the projection is degenerate at 0. Later, we will say that this projection ramifies at 0 and is unramified at every other $0 \neq a \in \Bbbk$.

Definition 1.14. Let A be an arbitrary ring. Take $\mathrm{Spec}(A) = \{P \subseteq A \mid P \text{ is a prime of } A\}$, the *spectrum* of A. We also define the usual topology on $\mathrm{Spec}(A)$: the so called basic opens are

$$D(f) := \{P \in \mathrm{Spec}(A) \mid f \notin P\}$$

One can check that $D(f) \cap D(g) = D(fg)$ hence it is a topological basis. The induced topology is called the *Zariski topology* of $\mathrm{Spec}(A)$.

Reminder: The localization $A[S^{-1}]$ of a ring A at a multiplicatively closed subset S (i.e. $x, y \in S$ implies $xy \in S$ and $1 \in S$ but $0 \notin S$) is defined as the unique ring with the universal property that the following diagram can always be completed in a unique way:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & A[S^{-1}] \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \exists!} \\ & & B \end{array}$$

for any morphism φ such that $\varphi(s)$ is a unit in B for all $s \in S$. For the construction, see any book on the basics of commutative algebra.

Remark 1.15. Observe that $P \in \mathrm{Spec}(A)$ and $P \cap S = \emptyset$ then $\rho(P)A_S \in \mathrm{Spec}(A_S)$.

Fact 1.16. *This induces a bijection*

$$\{P \in \mathrm{Spec}(A) \mid P \cap S = \emptyset\} \leftrightarrow \mathrm{Spec}(A_S)$$

Example 1.17. Two important special cases when we typically take localization is when $S_f = \{f^k \mid k \in \mathbb{N}\}$ and when $S_P = A \backslash P$ for some prime ideal P. We will use the notations $A_f := A[S_f^{-1}]$ and $A_P := A[S_P^{-1}]$.

Goal: There exists a sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Spec}(A)$ such that for all basic open set $D(f) \subseteq \mathrm{Spec}(A)$ we have $\mathcal{O}_X\big(D(f)\big) = A_f$. Moreover, for all $p \in X$, $\mathcal{O}_{X,p} = A_p$.

For this to make sense (i.e. to have the obvious restriction maps), we have to check the following lemma.

Lemma 1.18. *If $D(f) \subseteq D(g)$ then the image of g in A_f is invertible, i.e. there exists a factorization*

$$\begin{array}{ccc} A & \longrightarrow & A_g \\ & \searrow & \downarrow \\ & & A_f \end{array}$$

where the maps are the localization maps.

Proof. If g is not invertible in A_f then there exists a maximal ideal Q such that $Q \subseteq A_f$ containing g. Hence, there exists a prime ideal $P \in \operatorname{Spec}(A)$ such that $Q = PA_f$ by the bijection mentioned in Fact 1.16. Then $f \notin P$ but $g \in P$ which contradicts $D(f) \subseteq D(g)$ meaning that any prime ideal containing g must contain f too. □

Lemma 1.19. *Let X be a topological space and let $\mathcal{V}$ be a basis of open sets on X. Assume that for each $V \in \mathcal{V}$ there is a fixed abelian group $\mathcal{F}(V)$ and for all open inclusion, $V' \subseteq V$ a morphism $\rho_{VV'} : \mathcal{F}(V) \to \mathcal{F}(V')$ satisfying $\rho_{VV} = \mathrm{id}_{\mathcal{F}(V)}$ and $\rho_{VV''} = \rho_{V'V''} \circ \rho_{VV'}$ for all choice of open sets $V'' \subseteq V' \subseteq V$ in $\mathcal{V}$. Then*

1. *there exists a (not necessarily unique) presheaf $\mathcal{F}$ on X such that for all $V \in \mathcal{V}$, $\mathcal{F}(V)$ is the given one.*
2. *If $\mathcal{F}(V)$'s satisfy the sheaf axioms for coverings of V by open sets in $\mathcal{V}$ then the $\mathcal{F}$ above is unique and it is a sheaf.*
3. *Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X and for all $V \in \mathcal{V}$ maps $\varphi_V : \mathcal{F}(V) \to \mathcal{G}(V)$ such that*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

for all $U \subseteq V$ in $\mathcal{V}$, then there exists a unique morphism of sheaves $\mathcal{F} \to \mathcal{G}$ inducing these data.

Remark 1.20. The input of the lemma can be understood as some kind of a presheaf but only on a basis of the topology. Naturally, we will later apply this lemma on $\{D(f) \mid f \in A\}$.

Proof. For the completely detailed proof, see the online notes. Sketch: 1) It is done by an inverse limit construction over the basic open sets. Let $U \subseteq X$ be an open subset, not necessarily in $\mathcal{V}$. Then, we define

$$\mathcal{F}(U) := \big\{(f_V) \in \prod_{V \in \mathcal{V},\ V \subseteq U} \mathcal{F}(V) \mid \text{if } V' \subseteq V\ (f_V)|_{V'} = f_{V'}\big\}$$

Note that it is a non-filtered inverse limit. If $U \in \mathcal{V}$ then $\mathcal{F}(U)$ is the one given before. There exist "tautological" restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$. One can check that it is indeed a presheaf.

The second and third part of the lemma are straightforward checks of the definitions. □

Lemma 1.19 implies Goal 1.2. By the lemma, we only have to check that $D(f) \mapsto A_f$ satisfies the sheaf axioms. For the first axiom, suppose that $D(f) = \cup_i D(f_i)$, we have to prove that given $s \in \mathcal{O}_X\big(D(f)\big)$ such that $s|_{D(f_i)} = 0$ for all i then $s = 0$. In other words, we have to prove that the product $A_f \to \prod A_{f_i}$ of the localization maps defined in Lemma 1.18 is injective. We reduce to the case $f = 1$ by replacing A by A_f.

Given $s \in A$ such that $s \mapsto 0$ for all A_{f_i} means that there is an $n_i \in \mathbb{N}$ such that $f_i^{n_i} s = 0$ in A. Note that $D(f_i) = D(f_i^{n_i})$. Hence, $\cup D(f_i^{n_i}) = \cup D(f_i) = \operatorname{Spec}(A)$ by the assumption. It is equivalent to the fact that $f_i^{n_i}$ generate the whole ring, else there woul be a maximal (hence prime) ideal containg all of them. It means that there exist $g_i \in A$ such that $\sum g_i f_i^{n_i} = 1$. However, by the choice of n_i, it means that $s = \sum g_i f_i^{n_i} s = 0$. That was the statement.

The second sheaf axiom is a similar calculation. Our goal follows. The moreover part is a consequence of the fact $A_P = \lim_{\to f \notin p} A_f$ which is the same as the stalk of A at P. □

Definition 1.21. An *affine scheme* is a pair $(X, \mathcal{O}_X)$ defined as above.

Second Lecture, 21th of January

Reminder: Last time, along the way to define the notion of affine scheme, we defined the spectrum of a commutative unital ring A as a topological space. In fact, we proved that there exists a unique sheaf $\mathcal{O}_X$ on the spectrum of A such that the section on each $D(f)$ for $f \in A$ is the localization A_f. See Goal 1.2.

1.3 Schemes

Definition 1.22. A *ringed space* is a pair $(X, \mathcal{F})$ where X is a topological space, $\mathcal{F}$ is a sheaf of rings on X. A morphism of ringed spaces is $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(\varphi, \varphi^\sharp)$ where $\varphi : X \to Y$ is a continuous map and $\varphi^\sharp : \mathcal{G} \to \varphi_*\mathcal{F}$ is a morphism of sheaves. The notation $\varphi_*\mathcal{F}$ stands for the push-forward sheaf of $\mathcal{F}$ from X to Y defined as $\varphi_*\mathcal{F}(U) := \mathcal{F}\big(\varphi^{-1}(U)\big)$ for each $U \subseteq Y$. This way, ringed spaces form a category.

Remark 1.23. The rough meaning of it is that a map is a morphism if one can pull back sections along it.

Example 1.24. An affine scheme is a ringed space.

Observation: Given a morphism of ringed spaces as above, and $p \in X$ then $\varphi^\sharp$ induces a ring homomorphism $\varphi^\sharp_p : \mathcal{G}_{\varphi(p)} \to \mathcal{F}_p$. Indeed, we may take

$$\mathcal{G}_{\varphi(p)} = \varinjlim_{U \ni \varphi(p)} \mathcal{G}(U) \xrightarrow{\varphi^\sharp} \varinjlim_{U \ni \varphi(p)} \mathcal{F}\big(\varphi^{-1}(U)\big) \longrightarrow \varinjlim_{V \ni p} \mathcal{F}\big(V\big)$$

where $\{\varphi^{-1}(U) \subseteq X \mid \varphi(p) \in U\} \subseteq \{V \subseteq X \mid p \in V\}$ induces the second map.

Remark 1.25. Recall that a homomorphism of local rings $(A, M_A) \to (B, M_B)$ is *local* if $\varphi^{-1}(M_B) = M_A$.

Definition 1.26. A *locally ringed space* is a ringed space $(X, \mathcal{F})$ such that for all $p \in X$, $\mathcal{F}_p$ is a local ring. A morphism of locally ringed spaces is a morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ such that for all $p \in X$, $\varphi^\sharp_p$ is a local homomorphism.

Definition 1.27. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open covering $\{U_i\}_{i \in I}$ of X such that for all i, $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic (as a locally ringed space) to some affine scheme. Morphisms of schemes are defined to be the morphisms of locally ringed spaces. (In other words, Schemes is a full subcategory of the category of locally ringed spaces.)

Observation: There exists a contravariant functor Spec : CRings → Schemes (CRings is the category of commutative unital rings). On the objects, it is defined by $A \mapsto \mathrm{Spec}(A)$, and on morphisms, it is given by

$$(A \xrightarrow{\varphi} B) \mapsto \big(\mathrm{Spec}(\varphi), \mathrm{Spec}(\varphi)^\sharp\big)$$

where $\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $p \mapsto \varphi^{-1}(p)$ (It is indeed continuous as $\big(\mathrm{Spec}(\varphi)\big)^{-1}(D(f)) = D(\varphi(f))$.), and

$$\mathrm{Spec}(\varphi)^\sharp : \mathrm{Spec}(A) \to \varphi_*\mathrm{Spec}(B)$$

is defined on a basic open set $D(f) \subseteq \mathrm{Spec}(A)$ by the localization $\varphi_f : A_f \to B_{\varphi(f)}$ obtained as $A \xrightarrow{\varphi} B \to B_{\varphi(f)}$ factors through A_f. By Lemma 1.19/3, it is enough to define a morphism on a basis, it extends uniquely to the whole sheaf. This induces a local(!) homomorphisms on the stalks.

Theorem 1.28. *The contravariant functors $A \mapsto \operatorname{Spec}(A)$, $X \to \mathcal{O}_X(X)$ induce an anti-isomorphism of categories* CRings $\to$ AffSchemes *where* AffSchemes *is the full subcategory of schemes spanned by affine schemes, meaning that it is not only an antiequivalence (i.e. a contravariant equivalence functor), but really an (anti)isomorphism of categories.*

Proof. Clearly, on the objects, we have $A \mapsto \operatorname{Spec}(A) \mapsto \mathcal{O}_X\big(\operatorname{Spec}(A)\big) = A$, $X \mapsto \mathcal{O}_X(X) \to \operatorname{Spec}\mathcal{O}_X(X) = X$. Moreover, in the direction

$$(A \to B) \mapsto \big(\operatorname{Spec}(B) \to \operatorname{Spec}(A)\big) \mapsto (A \mapsto B)$$

it is straightforward to check that we get back the original map. In the reverse direction, it is not clear that in

$$\big(\operatorname{Spec}(B) \overset{(\varphi,\varphi^\sharp)}{\to} \operatorname{Spec}(A)\big) \mapsto (A \overset{\lambda}{\to} B) \mapsto \big(\operatorname{Spec}(B) \to \operatorname{Spec}(A)\big)$$

we get back $(\varphi, \varphi^\sharp)$ on the right. The basic question is that for $p \in \operatorname{Spec}(B)$, is it true that $\varphi(p) = \lambda^{-1}(p)$. By definition of λ, we have (localization) maps completing the following commutative square:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \downarrow & & \downarrow \\ A_{\varphi(p)} & \xrightarrow{\varphi_p^\sharp} & B_p \end{array}$$

Now, we use the assumption that $\varphi_p^\sharp$ is a local homomorphism, i.e. $\varphi(p)A_{\varphi(p)} = \big(\varphi_p^\sharp\big)^{-1} pB_p$. This ensures that the localizing prime ideal in A corresponds to the localizing prime ideal in B by λ. Therefore, $\lambda^{-1}(p) = \varphi(p)$. The other property we have to check is that

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ \downarrow & & \downarrow \\ A_f & \xrightarrow{\varphi_{D(f)}^\sharp} & B_{\lambda(f)} \end{array}$$

is commutative using the localization maps, proving $\varphi^\sharp = \operatorname{Spec}(\lambda)^\sharp$. ◻

Theorem 1.29. (Generalization) *Given a scheme X and a ring A, $Y \mapsto \mathcal{O}_Y(Y)$ induces a bijection* $\operatorname{Hom}\big(X, \operatorname{Spec}(A)\big) \cong \operatorname{Hom}\big(A, \mathcal{O}_X(X)\big)$. *The proof is analogous.*

Proof. We construct an inverse to $Y \mapsto O_Y(Y)$. Assume given $A \to \mathcal{O}_X(X)$ and let $U = \operatorname{Spec}(B) \subseteq X$ an affine open subset. First, we can compose the map with the restriction of $U \hookrightarrow X$ giving $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) = B$. Then given $f \in B$ we can form the composition $A \to B \to B_f = \mathcal{O}_X(D(f))$. Using the previous theorem, we can take the corresponding morphism in Schemes, giving $D(f) \hookrightarrow U \overset{\varphi^U}{\to} \operatorname{Spec}(A)$.

We have to prove that this way we get a well defined map $X \to \operatorname{Spec}(A)$ (i.e. a compatible set of maps). What happens if $W \subseteq V \cong \operatorname{Spec}(C) \subseteq X$? Then, we have a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & U \\ \uparrow & & \uparrow \\ V & \longleftarrow & W \end{array} \qquad \Rightarrow \qquad \begin{array}{ccc} \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_X(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_X(U) \end{array}$$

It means that $\varphi^U|_W = \varphi^V|_W$ for all basic open W contained in $U \cap V$, hence $\varphi^U|_{U\cap V} = \varphi^V|_{U\cap V}$ so they are compatible. Now, again by Lemma 1.19, we can patch these morphisms together. Now, we should only that it is indeed an inverse of $Y \mapsto \mathcal{O}_Y(Y)$. ◻

Example 1.30. Let $\Bbbk$ a field.

1. $\mathrm{Spec}(\Bbbk)$ is a 1-point space such that $\mathcal{O}_{\Bbbk}(\Bbbk) = \Bbbk$ which is also the stalk of this point.

2. The ring of dual numbers is defined as $\Bbbk[\varepsilon] = \Bbbk[x]/(x^2)$. Then $\mathrm{Spec}(\Bbbk[\varepsilon])$ is also a 1-point space but the stalk is $\Bbbk[\varepsilon]$. The natural morphism $\Bbbk[\varepsilon] \to \Bbbk$ induces a morphism $\mathrm{Spec}\Bbbk \to \mathrm{Spec}\Bbbk[\varepsilon]$ which is a homeomorphism but not an isomorphism of schemes.

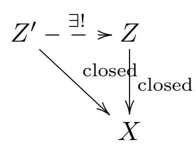
3. The spectrum $\mathrm{Spec}\Bbbk[t_1, \dots, t_n] =: \mathbb{A}^n_{\Bbbk}$ defines the affine space, where now $\mathbb{A}^n_{\Bbbk}$ contains all the prime ideals, not only the points. We also have a morphism of rings $\Bbbk \to \Bbbk[t_1, \dots, t_n]$ inducing a morphism of schemes $\mathbb{A}^n_{\Bbbk} \to \mathrm{Spec}\Bbbk$.

   Definition 1.31. A *closed immersion* $X \hookrightarrow Y$ of schemes is a morphism $(\varphi, \varphi^\sharp)$ such that φ is the inclusion of a closed set and there exists an open covering $Y = \cup \mathrm{Spec}A_i$ such that $\mathrm{Spec}(B_i) := X \cap \mathrm{Spec}(A_i) \hookrightarrow \mathrm{Spec}(A_i)$ (in particular, we require that $X \cap \mathrm{Spec}(A_i)$ is affine) corresponds to a surjective map $A_i \to B_i$.

   Remark 1.32. If $\varphi : A \to B$ is a surjection with $\mathrm{Ker}(\varphi) = I$ then $\mathrm{Spec}(B) = \mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$ is a closed immersion.

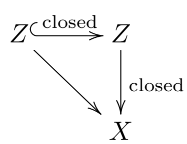
   Definition 1.33. A *closed subscheme* X of Y is a subscheme such that $X \hookrightarrow Y$ is a closed immersion.

   Remark 1.34. If X is a scheme and $Z \hookrightarrow X$ is a closed subset then there are several closed subscheme structures on Z. E.g. let $X = \mathrm{Spec}(A)$ and $Z = V(I) = V(I^2) = \cdots = V(I^n) =$ then $\mathrm{Spec}(A/I) \subseteq \mathrm{Spec}(A)$ is not necessarily equal to $\mathrm{Spec}(A/I^2)$ as schemes. But there is a terminal closed subscheme structure on Z, i.e. there exists a Z such that for all Z' the following commutative diagram can be completed

   (Not proven.) In the affine case, one can take $\mathrm{Spec}(A/\sqrt{I}) \to \mathrm{Spec}(A)$.

4. The affine closed set over $\Bbbk$ is a closed subscheme of $\mathbb{A}^n_{\Bbbk}$, i.e.

   Fact: (not proven) There always exists an ideal $I \subseteq A$ such that $Z \cong \mathrm{Spec}(A/I)$. In other words, for the affine space, all closed subschemes are, in fact, the good old affine algebraic varieties.

5. Scheme-theoretic inclusion of a point: Let X be a scheme and $p \in X$. Then $\mathcal{O}_{X,p}$ has a residue field $\kappa(p)$ defined by $\mathcal{O}_{X,p}/m_p$. We define $i_p : \mathrm{Spec}\kappa(p) \hookrightarrow X$ as follows:
   Suppose that $p \in U = \mathrm{Spec}(A) \subseteq X$. Then it corresponds to a prime ideal $P \subseteq A$. Hence, we have a surjection $\mathcal{O}_{X,p} = A_P \twoheadrightarrow \kappa(P)$. The associated morphism of $A \to A_P \to \kappa(P)$ is

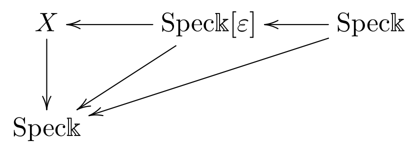
   $$\mathrm{Spec}\kappa(P) \to \mathrm{Spec}A_P \to \mathrm{Spec}A = U \hookrightarrow X$$

   defining i_p. One can check that this definition gives a well-defined map, i.e. the associated map of a different affine open set, the maps are compatible.

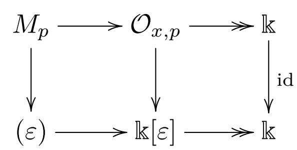
6. Generalization: Let R be a local ring, X be a scheme, $p \in X$, and $\mathcal{O}_{X,p} \to R$ a local homomorphism. Then we can construct (the same way as in Example 5.) a morphism $\mathrm{Spec}R \to X$. Obviously, for $R = \kappa(P)$ we get back Example 5.

7. If $X = \mathbb{A}^n_{\Bbbk}$, $p \in X$ and $\kappa(p) \cong \Bbbk$ then $i_p = \mathrm{Spec}\kappa(p) \to X$ corresponds to a morphism $\Bbbk[t_1, \dots, t_n] \to \Bbbk$, $x_i \mapsto a_i$, i.e. it corresponds to a $\Bbbk$-rational point of $\mathbb{A}^n_{\Bbbk}$. In fact, i_p is a section of the trivial map $\mathbb{A}^n_{\Bbbk} \to \mathrm{Spec}\Bbbk$.

8. Suppose we have $X \to \mathrm{Spec}\Bbbk$ and given morphisms making the following diagram commute



Then $\mathrm{Spec}\Bbbk \to X$ has the form i_p and so it corresponds to a point $p \in X$ where $\kappa(p) \cong \Bbbk$. Moreover, we have a local homomorphism $\mathcal{O}_{X,p} \to \Bbbk[\varepsilon]$ such that



where (ε) is $\Bbbk$ as a $\Bbbk$-module. Moreover, since $\varepsilon^2 = 0$, the map $M_p \to (\varepsilon)$ factors through $M_p/M_p^2 \to (\varepsilon) \cong \Bbbk$. As a conclusion, we get that maps $\mathrm{Spec}\Bbbk[\varepsilon] \to X$ (of $\Bbbk$-schemes) correspond to tangent vectors, i.e. element of $\mathrm{Hom}_{\Bbbk}(M_p/M_p^2, \Bbbk)$. This bijection can be generalized to the case when $\kappa(p)$ is not the base field but its separable finite extension. Then we have to write $\kappa(p)$ instead of $\Bbbk$ but the same statements are valid.

Definition 1.35. If X is a scheme and S is another scheme then a *point $p \in X$ with values in S* is just a morphism $S \to X$. Their set is denoted by $X(S) := \mathrm{Hom}_{\mathrm{Sch}}(S, X)$. Then $S \mapsto X(S)$ defines a contravariant functor $\mathrm{Sch} \to \mathrm{Sets}$ called the functor of points of X.

Proposition 1.36. *The functor h_X determines X up to unique isomorphism. (The proof is Yoneda's lemma.)*

Remark 1.37. In practice, one can obtain many information about a scheme before even knowing that they exist if we understand their functor of points. The fact the a functor is a functor of points of a scheme is called *representability* of the functor.

Definition 1.38. Let S be a fixed scheme. An *S-scheme* (or a schemes over S) is a morphism $X \to S$ and a morphism of S-schemes is a commutative triangle over S.

1.4 Patching schemes

Assume given a family of schemes $\{X_i \mid i \in I\}$ and for each $(i, j) \in I$ an open subscheme $U_{ij} \subseteq X_i$ (i.e. an open subset with the restricted structure scheme) such that $U_{ii} = X_i$. Moreover, assume given isomorphisms $\varphi_{ij} : U_{ij} \overset{\cong}{\to} U_{ji}$ satisfying $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ called cocycle condition (and also $\varphi_{ii} = \mathrm{id}_{U_{ii}}$ and $\varphi_{ij}^{-1} = \varphi_{ji}$). From these data one can construct a scheme X together with an open covering $\{U_i \mid i \in I\}$ such that $U_i \cong X_i$ and $U_{ij} \cong U_i \cap U_j$.

The underlying set of X is $\coprod X_i/\sim$ where $\sim$ identifies U_{ij} and U_{ji} via φ_{ij}. By the cocycle condition, it is transitive. Endow X with the quotient topology. The canonical open embedding $X_i \hookrightarrow X$ are denoted by

p_i. Consequently, we define $U_i := p(X_i)$. Then, we define $\mathcal{O}_X$ as follows: by Lemma 1.19, it suffices to define $\mathcal{O}_X(U)$ for open sets $U \subseteq U_i$ for all $i \in I$ (as they form a basis of the topology). The problem is that U may be a a subset of several U_i and we have to define the sheaf compatibly. So let

$$\mathcal{O}_X(U) := \left\{ (s_i)_{i \in I} \in \prod_{U \subseteq U_i} \mathcal{O}_{X_i}\big(p_i^{-1}(U)\big) \;\middle|\; \varphi_{ij}^\sharp(s_j) = s_i \right\}$$

i.e. we take the compatible sections for each i such that U_i contains U. In more abstract terms, this is a (non-filtered) inverse limit construction for the $\mathcal{O}_{X_i}\big(p_i^{-1}(U)\big)$'s and $\varphi_{ij}^\sharp$'s.

Example 1.39. Let A be a ring. We define $\mathbb{P}_A^n$ as a scheme. It is done by gluing together the affine schemes

$$D_+(x_i) = \mathrm{Spec} A\Big[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\Big]$$

. We also need the subsets along which we glue, i.e. define

$$D_+\Big(\frac{x_i}{x_j}\Big) = \mathrm{Spec} A\Big[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\Big]_{\frac{x_j}{x_i}} \subseteq D_+(x_i)$$

Notice that $D_+\Big(\frac{x_i}{x_j}\Big) \cong D_+\Big(\frac{x_i}{x_j}\Big) \subseteq A[x_0, \dots, x_n, x_0^{-1}, \dots, x_n^{-1}]$ giving us isomorphisms φ_{ij} i.e. a way to patch $D_+(x_i)$ with $D_+(x_j)$. One can check that these satisfy the cocycle condition and it agrees with the usual construction of the projective space.

Example 1.40. Special case for the projective space: $n = 1$. We glue two copies of $\mathrm{Spec}A[x]$ together along the maps $A[x] \to A[x, x^{-1}]$, $x \mapsto x$ and $A[x] \to A[x, x^{-1}]$, $x \mapsto x^{-1}$. Note, however, that we could try to do it in a wrong way, by using $A[x] \to A[x, x^{-1}]$, $x \mapsto x$ on both $\mathrm{Spec}A[x]$. The result of that would be the line with the doubled origin. Although this is a scheme, it is non-separated (see later) and it is definitely not the projective line.

Third Lecture, 28th of January

1.5 Fibre product

Definition 1.41. Let X, Y and Z be schemes and consider a commutative diagram of the following shape:

$$\begin{array}{ccc} & & Z \\ & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Then we may define their fibre product $Y \times_X Z$ which is a scheme whose functor of points is isomorphic to

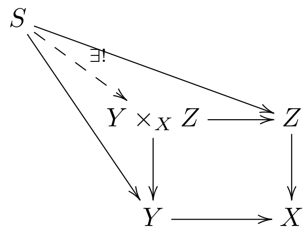
$$F_{YZ} : \text{Schemes} \longrightarrow \text{Sets}$$

that associates to S all the commutative squares

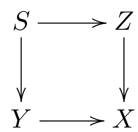
$$\begin{array}{ccc} S & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

together with the obvious morphisms of these "spans over X" (i.e. commutative diagrams of five objects in the appropriate arrangement).

Theorem 1.42. *There exists a scheme $Y \times_X Z$ over X such that F_{XY} is isomorphic to the functor of points of $Y \times_X Z$. In other words, for any S, any commutative diagram of the following shape can be uniquely completed:*

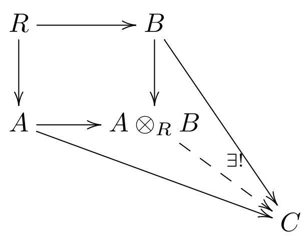


Lemma 1.43. *Assume given affine schemes $X = \mathrm{Spec}R$, $Y = \mathrm{Spec}A$, $Z = \mathrm{Spec}B$ together with morphisms $Y \to X \leftarrow Z$. Then the functor $S \mapsto$*set of comm. diagrams of the form



restricted to the category of affine schemes is isomorphic to the functor of points of the affine scheme $\mathrm{Spec}(A \otimes_R B)$.

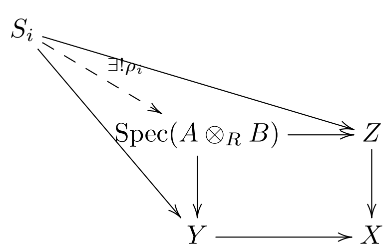
Proof. Using the anti-isomorphism of categories AffineSchemes $\leftrightarrow$ CommRings the requirement of the setup translates to



This holds (i.e. there exists a unique morphism for all C completing the diagram) by defining property of $A \otimes_R B$ i.e. the tensor product of R-algebras is the coproduct in the category of commutative R-algebras. □

Proof of Theorem 1.42. Assume X, Y and Z are affine as in the lemma. We claim that the functor of points of $\mathrm{Spec}(A \otimes_R B)$ is isomorphic to F_{YZ}, i.e. we now have to prove that for S being any scheme, the stated commutative diagram can be completed.

Choose an affine open covering $S = \cup_i S_i$. Then, by Lemma 1.43, for all i we have a unique commutative diagram



Similarly, for every affine subset $U \subseteq S_i \cap S_j$ the restrictions agree: $\rho_i|_U = \rho_j|_U$ by the uniqueness of Lemma 1.43, hence they glue together. Therefore, $F_{YZ}(U) \cong (Y \times_X Z)(U)$ for all U so the functors agree (on all objects, but one can prove that it is an isomorphism of functors), so they are the same by Yoneda's lemma. Also, ρ is unique, by Yoneda's lemma again.

For the next level of generality, assume that $X = \mathrm{Spec}(R)$ is still affine but let Y and Z be arbitrary schemes. Take affine open coverings $Z = \cup_{j\in J} Z_j$, $Y = \cup_{i\in I} Y_i$. We know that $Y_i \times_X Z_j$ exists for all i, j. First, fix $l \in J$. Note that, if $Y \times_X Z$ exists (i.e. it is representable) and $U \subseteq Y$ is open then F_{UZ} is the functor of points of $\pi_1^{-1}(U)$. First, we know that $Y_i \times_X Z_l$ is representable for fixed $l \in J$ and for all $i \in I$. Then, by the above mentioned note, $(Y_i \cap Y_j) \times_X Z_l$ also exists (i.e. the functor is representable) and is unique as $Y_i \cap Y_j$ is open. Moreover, $Y_i \supseteq Y_i \cap Y_j \cong Y_j \cap Y_i \subseteq Y_j$ hence we can get a unique morphism

$$Y_i \times_X Z_l \supseteq (Y_i \cap Y_j) \times_X Z_l \overset{\exists! \varphi_{ij}}{\leftrightarrow} (Y_j \cap Y_i) \times_X Z_l \subseteq Y_j \times_X Z_l$$

by uniqueness φ_{ij} must satisfy the cocycle condition and hence $Y_i \times_X Z_l$ patch together along $Y_{ij} \times_X Z_l$ so we get $Y \times_X Z_l$. We can do the same for the $Y \times_X Z_l$'s so we also get $Y \times_X Z$. Finally, assume X is general, and take an affine open covering $X = \cup X_i$, they will patch together to give $Y \times_{X_i} Z$. □

Remark 1.44. Note that, in general, the topological space of $Y \times_X Z$ is not equal to the fibre product of topological spaces. For example, take $X = \mathrm{Spec}\Bbbk$, $Y = \mathrm{Spec}L$ for some nontrivial finite field extension $\Bbbk \mid L$ and $Z = \mathrm{Spec}\overline{\Bbbk}$. Topologically, their fibre product is the one point space. While, the scheme theoretic fibre product is $Y \times_X Z = \mathrm{Spec}(L \otimes_{\Bbbk} \overline{\Bbbk})$ where

$$L \otimes_{\Bbbk} \overline{\Bbbk} = \big(\Bbbk[x]/(f)\big) \otimes_{\Bbbk} \overline{\Bbbk} = \overline{\Bbbk}[x]/\prod_i (x - \alpha_i) = \prod_{i=1}^{r} \overline{\Bbbk}$$

Another example is that $\mathbb{A}^2_{\Bbbk} = \mathbb{A}^1_{\Bbbk} \times_{\mathrm{Spec}(\Bbbk)} \mathbb{A}^1_{\Bbbk}$ but the topology of $\mathbb{A}^2_{\Bbbk}$ is not the product topology.

1.6 Fibres of a morphism

Definition 1.45. Let X and Y be schemes, $p \in X$ and consider a morphism, $Y \to X$. Then we can take the closed immersion $i_p : \mathrm{Spec}\,\kappa(p) \to X$. The *fibre* of φ at p is defined as the fibre product $Y \times_X \mathrm{Spec}\kappa(p)$.

Example 1.46. In the homework, we will compute the fibres of $\mathbb{A}^1 \to \mathbb{A}^1$, $z \mapsto z^n$.

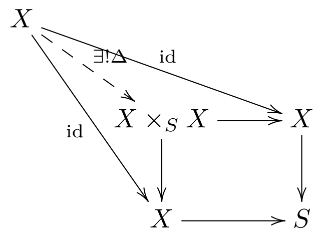
Proposition 1.47. *If $\varphi : Y \to X$ is a morphism and $p \in X$ then $\varphi : Y \times_X \mathrm{Spec}\kappa(p)$ as a topological space is homeomorphic to $\varphi^{-1}(p)$.*

Sketch of the proof. We may assume that X and Y are affine since p has an affine open neighborhood. So let $Y = \mathrm{Spec}B$, $X = \mathrm{Spec}A$ and $P \subsetneq A$ be the corresponding prime ideal. The morphism $Y \overset{\varphi}{\to} X$ induces a map $\lambda : A \to B$ that has a quotient $\overline{\lambda} : A/P \to B/\lambda(P)$. For a point $Q \in \mathrm{Spec}B$, $Q \in \varphi^{-1}(P)$ is equivalent to $Q \supseteq \lambda(P)$ and $\overline{\lambda}^{-1}(\overline{Q}) = (0)$. (If we would speak about maximal ideals then the second condition would be superfluous.)

This condition is equivalent to $\overline{\lambda}(A/P) \cap \overline{Q} = 0$ and $Q \supseteq \lambda(P)$. So for the choice $S := \overline{\lambda}(A/P) \backslash \{0\} \subseteq B/\lambda(P)B$, this is equivalent to the fact that $\overline{Q}$ defines a prime ideal in $\big(B/\lambda(P)B\big)_S$ (by Fact 1.16). However, $\big(B/\lambda(P)B\big)_S \cong B \otimes_A \kappa(P)$ as tensor product commutes with taking quotient and localizing. Therefore, prime ideals in $B \otimes_A \kappa(P)$ correspond to points of $\varphi^{-1}(P)$. □

1.7 Properties of schemes

Definition 1.48. For a scheme X over S (i.e. we have a fixed morphism $X \to S$) then the *diagonal* $\Delta = \Delta_{X/S} : X \to X \times_S X$ is the unique map completing the following commutative diagram:



Definition 1.49. X is *separated over* S if $\Delta_{X/S}$ is a closed immersion, and similarly, X is *separated* if it is separated over Spec$\mathbb{Z}$, the terminal object of the category.

Remark 1.50. Although, it has no precise effect (as the product of schemes can have different topological space than the product of their topological spaces), it is important to note that if T is a topological space, $T \stackrel{\Delta}{\to} T \times T$ is a closed immersion if and only if T is Hausdorff. Hence, separatedness is a generalization of the Hausdorff property.

Example 1.51.

- Any affine scheme is separated, and also any morphism of affine schemes is separated. Indeed, the diagonal map $\Delta_{X/S} : X \to X \times_S X$ for $X = \text{Spec}B$ and $S = \text{Spec}A$ induces the multiplication map $B \otimes_A B \to B$, $(x, y) \mapsto xy$. It is clearly, surjective, hence $\text{Spec}(B) \to \text{Spec}(B \otimes_A B)$ is a closed immersion.

- A non-example: $\mathbb{A}^1$ with a doubled origin is not separated. (It is a not-immediate exercise to prove it.)

Proposition 1.52. *Let X be a scheme. The following are equivalent:*

1. *X is separated*

2. *For any two $U, V \subseteq X$ affine open subschemes, $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ (the multiplication of restrictions) is surjective.*

3. *There exists an affine open covering $X = \cup U_i$ such that the $U_i \cap U_j$ satisfy* (2)*.*

Proof. One can identify the intersection $U \cap V \cong \Delta(X) \cap (U \times V)$ where the latter is interpreted in $X \times X$. This shows that 1) implies 2) as Δ is closed immersion by assumption and $U \times V$ is an affine scheme, but a closed subscheme of an affine scheme is also affine. Implication 2) $\Rightarrow$ 3) is clear. To see the last one 3) $\Rightarrow$ 1), note that $\Delta(X) \cap (U_i \cap U_j)$ is a closed immersion in $U_i \times U_j$ for all i, j hence Δ glue together to be a closed immersion as it is a local property. □

Corollary 1.53. $\mathbb{P}^n_{\mathbb{Z}}$ *is separated by* 3) *applied on the standard open covering. Also,* $\mathbb{P}^n_A$ *is separated over* Spec$A$ *for all ring A as* $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec}\mathbb{Z}} \text{Spec}A$*. Moreover, quite generally, if $Y \to X$ is separated then $Y \times_X Z \to Z$ is separated for all $Z \to X$. (One can prove it by the general argument method by taking affine cover. Then the statement melts down to prove that if $A \to B$ is surjective then $A \otimes C \to B \otimes C$ is also surjective.)*

Definition 1.54. $\varphi : Y \to X$ is *proper* if

- φ is separated

- φ is of finite type
- for all $Z \to X$, $Y \times_X Z \to Z$ is a closed mapping. (This change: $(Y \to X) \rightsquigarrow (Y \times_X Z \to Z)$ is called a base-change.)

Where φ is *of finite type* if there exists a finite affine open covering $X_i = \text{Spec}A_i$ such that $\varphi^{-1}(\text{Spec}A_i) = \cup_j \text{Spec}B_{ij}$ for some rings B_{ij} where B_{ij} is a finitely generated A_i-algebra.

Similarly, φ is called *finite* using the same definition as above but requiring that B_{ij} is a finite module over A_i.

Example 1.55.

1. A finite morphism is proper. (Exercise. Its affine case, in particular the "every base change is closed" property depends on the going up theorem.)
2. Projective morphisms are proper. Here, a morphism $\varphi : Y \to X$ is called *projective* if there exists a factorization.

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \\ & {\scriptstyle\varphi}\searrow & \downarrow \\ & & X \end{array}$$

Note that we have basically seen the special case of this in the last semester in the form of the Main theorem of elimination theory.

2 Quasi-coherent sheaves

Definition 2.1. Let X be any scheme. A *sheaf of $\mathcal{O}_X$-modules*, (or in short, an $\mathcal{O}_X$-module) is a sheaf of abelian groups $\mathcal{F}$ on X such that for all $U \subseteq X$ is open, $\mathcal{F}(U)$ has an $\mathcal{O}_X(U)$-module structure compatible with the restriction maps, i.e. for open sets $V \subseteq U$ the restriction maps commute with the structure morphisms:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

Example 2.2.

1. For a morphism $\varphi : X \to Y$ one can define the associated $\varphi^\sharp : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ where $\varphi_*\mathcal{O}_X$ is trivially a sheaf of $\mathcal{O}_Y$-modules.
2. In this situation, let $\mathcal{I} = \text{Ker}(\varphi^\sharp)$ i.e. $\mathcal{I}(U) := \text{Ker}\big(\mathcal{O}_Y(U) \to \varphi_*\mathcal{O}_X(U)\big) \subseteq \mathcal{O}_Y(U)$ then $\mathcal{I}$ is a sheaf of $\mathcal{O}_Y$-modules. In fact, a sheaf of ideals.

Definition 2.3. Let A be a ring and M an A-module. If $S \subseteq A$ is a multiplicatively closed subset then we can localize it as $M_S := M \otimes_A A_S$. Then we have a natural morphism $A \to A_S$ and the corresponding map (by tensoring by M) of modules $M \to M_S$. In particular, if $f \in A$ then we have an associated morphism $M \to M_f$ and if $D(f) \subseteq D(g)$ then we have $M_g \to M_f$ as we have seen in Lemma 1.18. The sheaf axioms for $\mathcal{O}_X$ imply that $\{M_f\}_{f \in A}$ also satisfy the sheaf axioms, hence – by Lemma 1.19 – we get a sheaf $\tilde{M}$ on X which is an $\mathcal{O}_X$-module as the M_f's are modules over A_f.

We can extend this map to get a functor $\tilde{\ } : A\text{−mod} \longrightarrow \mathcal{O}_X\text{−modules}$ that is

- fully faithful, i.e. $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M},\tilde{N})$ is a bijection (by taking global sections)
- exact, i.e. it brings exact sequences to exact sequences, as A_f is a flat A-module, i.e. tensoring with A_f is exact and exactness of sheaves on every basic open subset is a stronger property than being exact as a sheaf of abelian groups.

Example 2.4. It is not true that this functor is essentially surjective i.e. that every $\mathcal{O}_X-$module over $X = \mathrm{Spec}(A)$ is of the form $\tilde{M}$ for some A-module M. E.g. take A to be a DVR like $A = \Bbbk[x]_{(x)}$. Then $\mathrm{Spec}A$ has two points, corresponding to (0) and the maximal ideal m. Then (0) is an open point whose closure is X while m is a closed point. So $\mathrm{Spec}A$ has two nonempty open subsets X and $\{\eta\}$. To define an $\mathcal{O}_X$-module, it is enough to define $\mathcal{F}(X) \to \mathcal{F}(\{\eta\})$. Let $\mathcal{F}(X) = A$ and $\mathcal{F}(\{\eta\}) = 0$. Then $\mathcal{F}$ is not of the form $\tilde{M}$ for some M as we must have $M = \mathcal{F}(X) = A$ but $A_m = \mathcal{O}_X(\{\eta\}) = \mathrm{Frac}(A)$ which is clearly nonzero.

Definition 2.5. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasi-coherent sheaf* on X if there exists an affine open covering $X = \cup_i \mathrm{Spec}A_i$, $\mathcal{F}|_{\mathrm{Spec}A_i} \cong \tilde{M}_i$ for some A_i-mod, M_i. If, moreover, M_i is finitely generated over A_i then $\mathcal{F}$ is a coherent sheaf.

Proposition 2.6. *The following are equivalent for an $\mathcal{O}_X$-module $\mathcal{F}$:*

1. *$\mathcal{F}$ is quasi-coherent,*
2. *For all affine open $U = \mathrm{Spec}A \subseteq X$ there is an A-module M such that $\mathcal{F}|_U \cong \tilde{M}$,*
3. *There exists an open covering $X = \cup_i U_i$ such that for all i, $\mathcal{F}|_{U_i} \cong \mathrm{Coker}\big(\mathcal{O}_{U_i}^{\oplus J} \to \mathcal{O}_{U_i}^{\oplus K}\big)$ for some index sets J and K.*

Corollary 2.7. *If $X = \mathrm{Spec}A$ then $M \mapsto \tilde{M}$ induces an equivalence of categories from $A-\mathrm{mod}$ to quasi-coherent sheaves on X.*

Corollary 2.8. *If $X = \mathrm{Spec}A$ is an affine scheme then an exact sequence of quasi-coherent sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ induces an exact sequence of abelian groups $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$.*

Proof. We only have to prove the surjectivity of $M := \mathcal{G}(X) \to \mathcal{H}(X) =: N$. Then, by Proposition 2.6 we have $\mathcal{G} = \tilde{M}$, $\mathcal{H} = \tilde{N}$. Set $P := \mathrm{Coker}(M \xrightarrow{\varphi} N)$. Since $M \mapsto \tilde{M}$ is an exact functor, we get that $\tilde{M} \to \tilde{N} \to \tilde{P} \to 0$ is exact, hence $\tilde{P} = 0$. □

Fourth Lecture, 4th of February

Proof of 2.6. 2) implies 3) is clear since we can take an affine open covering $\{U_i\}_{i\in I}$ on which we have $\mathcal{F}|_{U_i} = \tilde{M}_i$ for some $\mathcal{O}_X(U_i)$-module M_i. Then, as all A_i-modules (for now $A_i = \mathcal{O}_X(U_i)$ have a presentation (possibly infinite) $\mathrm{Coker}(A_i^{\oplus I} \to A_i^{\oplus J}) \cong M$. The corresponding cokernel gives a presentation for $\tilde{M}_i$.

To prove 3) implies 1), first, we refine the covering so that we may assume that $U_i = \mathrm{Spec}(A_i)$ is affine. Then, over U_i, we have that $\mathcal{F}|_{U_i} = \mathrm{Coker}\big(\mathcal{O}_{U_i}^{\oplus J} \to \mathcal{O}_{U_i}^{\oplus K}\big)$. Take the corresponding map of modules $A_i^{\oplus I} \to A_i^{\oplus J}$ and the corresponding cokernel $M_i := \mathrm{Coker}\big(A_i^{\oplus I} \to A_i^{\oplus J}\big)$. Then $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ so we got 1).

The hard direction is 1) implies 2). Assume that $X = \cup_i U_i$ where $U_i = \mathrm{Spec}(A_i)$ are the affines where $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for some A_i-modules M_i. First, we deduce that $\mathcal{F}|_{D(f)}$ also comes from a module, for any $f \in A_i$ and $i \in I$. As we have $D(f) \subseteq U_i$ we can prove that $\mathcal{F}|_{D(f)} \cong \widetilde{M_i \otimes_{A_i} (A_i)_f}$. Indeed, if $D(g) \subseteq D(f)$ then f is a unit in $(A_i)_g$ hence $(A_i)_g$ is a further localization of $(A_i)_g$ by Lemma 1.18. This proves that we not only have $\mathcal{F}(D(f)) = M_i \otimes_{A_i} (A_i)_f$ which is clear but they are isomorphic as $\mathcal{O}_{D(f)}$-modules.

Now, let $U = \mathrm{Spec}(A) \subseteq X$ be open and consider its open covering $\{U \cap U_i\}_i$. One can write U as a union of basic affine open sets $\cup_j D(g_j)$ where $g_j \in \mathcal{O}(U_i)$ for some i. Then, define the sheaves $\mathcal{F}_j$, $\mathcal{F}_{jk}$ on U as

$$\mathcal{F}_j(V) := \mathcal{F}\big(V \cap D(g_j)\big) \qquad \text{and} \qquad \mathcal{F}_{jk}(V) := \mathcal{F}\big(V \cap D(g_j g_k)\big)$$

for any open subset $V \subseteq U$. One can check that these are sheaves. Moreover, by the sheaf axioms, we have $\mathcal{F}|_U = \mathrm{Ker}\Big(\prod_j \mathcal{F}_j \to \prod_{j\neq k} \mathcal{F}_{jk}\Big)$. Our goal is to patch together the modules on the basic open sets to a module on another affine open subset.

For all i, we have $\mathcal{F}|_{D(g_i)} \cong \tilde{N}_i$ for some A_{g_i}-module N_i by the first paragraph, hence $\mathcal{F}_i \cong \tilde{N}_i$ where N_i is viewed as an A-module via the embedding $A \to A_{g_i}$. Similarly, $\mathcal{F}_{ij} \cong \tilde{N}_{ij}$ for some A-module N_{ij}. In fact, it is an $A_{g_i g_j}$-module the same way as before. Now, let $N := \mathrm{Ker}\Big(\prod_i N_i \to \prod_{i\neq j} N_{ij}\Big)$. since $N \mapsto \tilde{N}$ is an exact functor, we get that

$$\tilde{N} = \mathrm{Ker}\Big(\prod_i N_i \to \prod_{i\neq j} N_{ij}\Big) = \mathcal{F}|_U$$

Hence we got the second statement. □

Definition 2.9. A *coherent sheaf* is a quasi-coherent sheaf where the A_i-modules M_i are also finitely generated.

Definition 2.10. A scheme X is *locally Noetherian* if it has an open covering $\cup_i \mathrm{Spec} A_i = X$ where A_i is a Noetherian ring for all $i \in I$. Moreover, X is *Noetherian* if there exists such a finite covering, equivalently, if it is locally Noetherian and quasi-compact.

Proposition 2.11. *(without proof) Let X be a locally Noetherian scheme. Then for any affine open subset $U \subseteq X$, U is also Noetherian.*

Remark 2.12. Note that it is not true if X is Noetherian then $\mathcal{O}_X(U)$ is Noetherian for all (not necessarily affine) open U. Moreover, requiring that every stalk $\mathcal{O}_{X,p}$ is Noetherian is not enough to ensure that X is locally Noetherian, e.g. take $\mathbb{F}_2^{\mathbb{N}}$.

Proposition 2.13. *(without proof) If X is locally Noetherian then the following are equivalent for an $\mathcal{O}_X$-module $\mathcal{F}$:*

1. *$\mathcal{F}$ is coherent,*
2. *For all $U = \mathrm{Spec}(A)$, $\mathcal{F}|_U = \tilde{M}$ for a finitely generated A-module M,*
3. *There exists an open covering $X = \cup_{i\in I} U_i$, $\mathcal{F}|_{U_i} \cong \mathrm{Coker}\big(\mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus s}\big)$ for all $i \in I$.*

Remark 2.14. The Noetherian property kicks in when we take a presentation for a finitely generated module: it is a quotient of a finitely generated free module but the kernel is not necessarily finitely generated without the Noetherian property.

2.1 Vector bundles

Let $\varphi : X \to Y$ be a morphism of schemes and consider the $\mathcal{O}_Y$-module $\varphi_* \mathcal{O}_X$.

Question 2.15. *Is this a quasi-coherent sheaf on Y?*

In general, no.

Definition 2.16. A morphism $\varphi : X \to Y$ of schemes is *affine* if there exists an affine open covering $Y = \cup_i \mathrm{Spec} A_i$ such that $\varphi^{-1}(U_i)$ is affine for all U_i.

Example 2.17. If we moreover require that $\mathcal{O}_{\varphi^{-1}(U_i)}\big(\varphi^{-1}(U_i)\big)$ is a finitely generated A_i-module, this is a finite morphism. A special case of this is an open immersion.

Proposition 2.18. *If $\varphi : X \to Y$ is affine then $\varphi_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module and $I_X := \mathrm{Ker}\big(\varphi^\sharp\mathcal{O}_Y \to \varphi_*\mathcal{O}_X\big)$ is also a quasi-coherent $\mathcal{O}_X$-module.*

Proof. By the assumption, we can reduce to the case $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$. Then $X \to Y$ corresponds to a morphism of rings $\lambda : A \to B$. Moreover, $\varphi_*\mathcal{O}_X$ corresponds to $\tilde{B}$ where B is an A-module via λ. Similarly, I_X corresponds to $\tilde{I} = \widetilde{\mathrm{Ker}(\lambda)}$. Hence, both are quasi-coherent. □

Corollary 2.19. *The above defined $(X \to Y) \mapsto I_X$ gives a bijection between closed subschemes and quasi-coherent shaves of ideals on Y.*

Proof. Assume $I_X \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals. Then, by definition, there exists an affine open covering $X = \cup_i \mathrm{Spec}(A_i)$ such that $I_X|_{\mathrm{Spec}(A_i)} \cong \tilde{I}_i$ for some ideals $I_i \subseteq A_i$. Consider the closed immersion $\mathrm{Spec}A_i|_{I_i} \hookrightarrow \mathrm{Spec}A_i$. One can check(!) that these patch together to a closed immersion $X \to Y$ using the argument of the proof of 1.29. □

Definition 2.20. A *quasi-coherent sheaf of $\mathcal{O}_X$-algebras* is a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-algebra for all $U \subseteq X$ open and $\mathcal{F}(U) \to \mathcal{F}(V)$ are algebra homomorphisms for all $V \subseteq U$. Equivalently, there exists an open covering $X = \cup_i \mathrm{Spec}A_i$ such that $\mathcal{F}|_{\mathrm{Spec}A_i} \cong \tilde{M}_i$ where M_i is an A_i-algebra.

Proposition 2.21. *Assume given a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-algebras. Consider the functor F on the category of schemes over X defined as follows:*

$$F : (Y \xrightarrow{\varphi} X) \mapsto \mathrm{Hom}(\mathcal{F}, \varphi_*\mathcal{O}_Y)$$

Then F is the functor of points of a scheme $\mathrm{Spec}(\mathcal{F}) \to X$. Moreover, $\mathrm{Spec}(\mathcal{F}) \to X$ is an affine morphism.

Example 2.22. For $\mathcal{F} = \mathcal{O}_X$ we get $\mathrm{Spec}(\mathcal{F}) = X$ with the identity morphism.

Sketch of the proof. Assume that $X = \mathrm{Spce}(A)$ and $\mathcal{F} = \tilde{M}$ where M is an A-algebra. Suppose $Y = \mathrm{Spec}(B)$ is also affine. Then

$$\mathrm{Hom}_{\mathcal{O}_X\text{-algebra}}(\mathcal{F}, \varphi_*\mathcal{O}_X) \cong \mathrm{Hom}_{A\text{-algebra}}(M, B)$$

where the isomorphism is realized by taking global sections. Since both sheaves are quasi-coherent, it is an isomorphism. However, here M and B are rings, hence

$$\mathrm{Hom}_{A\text{-algebra}}(M, B) \cong \mathrm{Hom}_X(Y, \mathrm{Spec}M)$$

by the anti-equivalent of affine schemes and rings. Generally, $\mathrm{Spec}(\mathcal{F})$ is defined by patching together $\mathrm{Spec}(M)$, as in the proof of 1.29. □

Definition 2.23. Let X be a scheme. A *locally free sheaf* on X is an $\mathcal{O}_X$-module $\mathcal{F}$ such that there exists an open covering $X = \cup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n_i}$ for some $n_i > 0$. Note that if X is connected the for all n_i are the same fixed n. This is called the *rank* $\mathrm{rk}\mathcal{F}$ of $\mathcal{F}$.

An open covering $X = \cup_i U_i$ as above is called a trivialization of $\mathcal{F}$.

Proposition 2.24. *A locally free sheaf is coherent. If, moreover, X is locally Noetherian and connected then a coherent sheaf $\mathcal{F}$ on X is locally free if and only if $\mathcal{F}_p$ is a free $\mathcal{O}_{X,p}$-module for all $p \in X$.*

Proof. If $U = \mathrm{Spec}(A) \subseteq U_i$ for some i then $\mathcal{F}|_U \cong \widetilde{A^{\oplus n_i}}$ hence it is coherent. The $\Rightarrow$ direction of the "moreover" part follows from the fact that $\mathcal{F}_p = \lim_\to \mathcal{F}(U)$ is U is so small that $U \subseteq U_i$ for some i then we get that $\lim_\to \mathcal{O}(U)^{\oplus n_i} = \mathcal{O}_p^{\oplus n_i}$.

Conversely, assume that $\mathcal{F}_p$ is free over $\mathcal{O}_{X,p}$ for all $p \in X$. Let p be fixed and denote the generators of $\mathcal{F}_p$ by $t_1, \dots, t_n$. For a sufficiently small open neighborhood U of p, $t_1, \dots, t_n$ generate $\mathcal{F}(U)$ over $\mathcal{O}(U)$. We may assume that $U = \mathrm{Spec}(A)$ where $\mathcal{F}|_U = \tilde{M}$. Then $t_1, \dots, t_n$ is a system of generators for M over A. By locally Noetherian property A is Noetherian. Hence, we get a short exact sequence

$$0 \longrightarrow K \longrightarrow A^{\oplus n} \longrightarrow M \longrightarrow 0$$

where K is finitely generated by the Noetherian assumption. The latter map induces an isomorphism when tensored with $\mathcal{O}_{X,p}$. Let $s \in K$ i.e. there exists an open neighborhood V of p such that $V \subseteq U$ and $s|_V = 0$. Since K is finitely generated, choosing a generating system $s_1, \dots, s_m$ for K, we find that there exists a V (the intersection of the good V's) such that $s_i|_V = 0$ for all i. Hence, $\tilde{K}|_V = 0$ meaning that $\mathcal{O}_U^{\oplus n}|_V \to \mathcal{F}|_V$ is an isomorphism. □

Exercise 2.25. (by the proof, it is easy) If $\mathcal{F}$ is a coherent sheaf that is locally Noetherian and $p \in X$ is a point such that $\mathcal{F}_p = 0$ then there exists an open neighborhood U of p such that $\mathcal{F}|_U = 0$.

Definition 2.26. A *vector bundle* $\mathbb{V} \xrightarrow{p} X$ is a scheme over X such that there exists an affine open covering $X = \cup_i U_i$, $U_i = \mathrm{Spec} A_i$ and isomorphisms $\mathbb{V} \times_X U_i \xrightarrow{\varphi_i} \mathbb{A}^n_{U_i} = \mathrm{Spec}\big(A_i[T_1, \dots, T_n]\big)$. Moreover, for all $U \subseteq \mathrm{Spec} A$ contained in $U_i \cap U_j$ the automorphism of $\mathbb{A}^n_U$ given by $\varphi_i^{-1}|_U \circ \varphi_j|_U$ corresponds to an A-linear automorphism of $A[T_1, \dots, T_n]$ (i.e. preserving the natural grading). Given an open subset U, a section of $\mathbb{V}$ over U is a morphism $s : U \to V$ such that $U \xrightarrow{s} V \xrightarrow{p} X$.

Proposition 2.27. *Given a rank n vector bundle $\mathbb{V} \to X$, the presheaf $U \mapsto \{\text{sections of } \mathbb{V} \to X \text{ over } U\}$ is a locally free sheaf on X. In this way, we obtain an equivalence of categories between vector bundles and $\mathbb{V} \to X$ and locally free sheaves.*

Proof. Assume $U \subseteq X$ is such that $U \times_X \mathbb{V} \cong \mathbb{A}^n_U$. Then we have

$$s_V(U) \cong \mathrm{Hom}_{A\text{-alg}}\big(A[t_1, \dots, T_n], A\big) \cong A^{\oplus n}$$

hence, we get that $s_V|_U \cong \mathcal{O}_U^{\oplus n}$ proving that it is indeed a locally free sheaf.

Conversely, assume given a locally free sheaf $\mathcal{F}$ on X. Assume that $U = \mathrm{Spec} A \subseteq X$ is so small that $\mathcal{F}|_U \cong \mathcal{O}_U^n$. Set $\mathbb{V}_U = \mathrm{Spec} A[T_1, \dots, T_n]$. We construct $\mathbb{V}$ by patching the $\mathbb{V}_U$ together. If we have an isomorphism $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$ and $\mathcal{F}|_V \cong \mathcal{O}_V^{\oplus n}$ then for $W = \mathrm{Spec} R \subseteq U \cap V$ the map $\varphi_U^{-1}|_W \circ \varphi_V|_W$ induces an R-linear automorphism of $R[T_1, \dots, T_n]$ as it is a module-homomorphism. Then, one has to prove that $\mathbb{V}$ patch together and that the two associations really give a bijection. □

Construction of $\mathbb{V}$ associated to $\mathcal{F}$, a'la Grothendieck. Recall that if A is a ring then the forgetful functor A-algebras $\to$ A-mod has a left adjoint $M \mapsto \mathrm{Sym}(M)$ called the symmetric algebra of M, i.e. it is the free A-algebra on M. It is defined as

$$\mathrm{Sym}(M) := T(M)/(m \otimes n - n \otimes m) = \oplus_{i=0}^{\infty} M^{\oplus i}/(m \otimes n - n \otimes m)$$

Observe that if $M = A^{\oplus n}$ then $\mathrm{Sym}(M) = A[T_1, \dots, T_n]$.

Remark 2.28. If $A \to C$ is a ring homomorphism then we have the base change property i.e. $\mathrm{Sym}(M) \otimes_A C = \mathrm{Sym}(M \otimes_A C)$.

By this remark, given a quasi-coherent sheaf $F = \tilde{M}$ on $\mathrm{Spec} A$ we may define $\mathrm{Sym}(\mathcal{F}) := \widetilde{\mathrm{Sym}(M)}$. This extends to an arbitrary X by passing to an affine open covering, i.e. the forgetful functor from quasi-coherent $\mathcal{O}_X$-algebras to quasi-coherent $\mathcal{O}_X$-modules on X has a left adjoint $\mathcal{F} \mapsto \mathrm{Sym}(\mathcal{F})$.

Definition 2.29. If $\mathcal{F}$ is a locally free sheaf on X then we can define $\mathcal{F}^\vee$ the sheaf given by $\mathcal{F}^\vee(U) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{O}_U)$. This is also a locally free sheaf as a dual of a free module is again free.

Now, we can define $\mathbb{V}$ as $\mathrm{Spec}\big(\mathrm{Sym}(\mathcal{F}^\vee)\big) \to X$ is the vector bundle associated with $\mathcal{F}$. W check that it is the same as before. If $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$ for $U = \mathrm{Spec} A$, $\mathrm{Sym}(\mathcal{F}^\vee)|_U \cong \mathrm{Hom}_A\big(A[T_1, \dots, T_n], A\big)$. Now, it is visible that the dual is needed so that we get the sections and not just the polynomial ring.

Fifth Lecture, 11th of February

2.2 Invertible sheaves

Definition 2.30. An *invertible sheaf* is a locally free sheaf of rank 1 i.e. it is an $\mathcal{O}_X$-module $\mathcal{L}$ such that any $p \in X$ has an open neighborhood U such that $\mathcal{L}|_U \cong \mathcal{O}_U$.

Remark 2.31. By the investigation we did last time, locally free sheaves of rank one correspond to vector bundles of rank one, that are called *line bundles.*

Definition 2.32. A scheme X is *integral* if for any open subset $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral scheme.

Lemma 2.33. *The underlying topological space of an integral scheme is irreducible.*

Proof. If there exists U_1 and U_2 open subsets of X such that $U_1 \cap U_2 = \emptyset$ then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2)$ is not an integral domain. □

Remark 2.34. (without proof) The converse with some modification is also true: If X is irreducible and $\mathcal{O}_X(U)$ has no nilpotents (i.e. it is reduced) for all open subsets $U \subseteq X$ then X is integral.

Proposition 2.35. *Let X be an integral scheme. Then*

1. *there exists a unique point called the generic point $\eta \in X$ such that X is the closure of $\{\eta\}$.*
2. *The stalk of $\mathcal{O}_{X,\eta}$ is a field K that is the fraction field of every local ring of X.*

Proof. First, we prove the uniqueness: if η_1 and η_2 are generic points then for any $X \supseteq U = \operatorname{Spec}(A)$ affine open has to contain η_1 and η_2 as they are dense points. It means that η_i corresponds to a prime idal $P_i \subseteq A$ such that $V(P_i) = \operatorname{Spec}(A)$ $(i = 1, 2)$. It can happen only for $P_1 = P_2 = (0)$ so they are the same, i.e. $\eta_1 = \eta_2$. Moreover, the existence is clear from the above observation: for any affine open $\operatorname{Spec}(A) = U \subseteq X$ we have that A is an integral domain hence (0) will be a generic point for X.

To prove the second part, notice that $\mathcal{O}_{X,\eta} = \operatorname{Frac}(A)$ for an affine open $\operatorname{Spec}(A) = U \subseteq X$ as above. It is also the common fraction field of all local rings of U. By the uniqueness of the generic point, we are done, as all points have an affine open neighborhood. □

Let X be an integral scheme.

Definition 2.36. Denote by $\mathcal{K}$ the constant sheaf on X defined by the field $K := \mathcal{O}_{X,\eta}$. It is an $\mathcal{O}_X$-module.

Proposition 2.37. *Every invertible sheaf $\mathcal{L}$ on X is isomorphic to a sub-$\mathcal{O}_X$-module of $\mathcal{K}$.*

Remark 2.38. The proof of this statement in Hartshorn's Algebraic Geometry is flawed.

Proof. Choose a nonempty, open subset $U \subseteq X$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Let $j : U \hookrightarrow X$ be the inclusion map. Then there exists a map of sheaves $\mathcal{O}_X \to j_*\mathcal{O}_U$ mapping $\mathcal{O}_X(V) \ni s \mapsto s|_{U\cap V}$ for any open subset $V \subseteq X$.
Claim 2.39. This map is injective.

Proof. If $V = \operatorname{Spec}(B) \subseteq X$ is an affine open subset and $D(f) \subseteq U \cap V$ a basic open subset for $f \in B$ then

$$B \cong \mathcal{O}_X(V) \ni s \mapsto s|_{U\cap V} \mapsto s|_{D(f)} \in \mathcal{O}_X\big(D(f)\big) \cong B_f$$

is injective as B is an integral domain. Hence, the same map is also injective for all basic affine opens $D(f) \subseteq U \cap V$, hence the claim. □

Corollary 2.40. *$\mathcal{L} \to j_*\mathcal{L}$ is also injective.*

Proof. Injectivity can be checked on the stalks

$$\begin{array}{ccc} \mathcal{L}_p & \longrightarrow & (j_*\mathcal{L})_p \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{O}_{X,p} & \hookrightarrow & (j_*\mathcal{O}_U)_p \end{array}$$

for all $p \in X$. ☐

Therefore, we have a chain of embeddings $\mathcal{L} \hookrightarrow j_*(\mathcal{L}|_U) \cong j_*\mathcal{O}_U \hookrightarrow j_*(\mathcal{K}|_U) \cong \mathcal{K}$ as $\mathcal{K}$ is constant and $U \subseteq X$ is dense. Hence, we get an embedding into the constant sheaf. ☐

Definition 2.41. Recall that if X is an integral scheme then a *Cartier divisor* on X is given by an open covering $\cup_i U_i = X$ and for all i, $f_i \in \mathcal{K}(U_i)^\times$ such that

$$f_i f_j^{-1}|_{U_i \cap U_j}, f_j f_i^{-1}|_{U_i \cap U_j} \in \mathcal{O}_X(U_i \cap U_j)$$

Two such systems $\big\{(U_i, f_i)\big\}, \big\{(V_j, g_j)\big\}$ define the same Cartier divisor (i.e. a Cartier divisor is an equivalence class modulo this equivalence relation) if $f_i g_j^{-1}|_{U_i \cap V_j}, f_i^{-1} g_j|_{U_i \cap V_j} \in \mathcal{O}_X(U_i \cap V_j)$ for all $i \neq j$.

Proposition 2.42. *There exists a bijection between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$.*

Proof. To a Cartier divisor $D = \big[(U_i, f_i)\big]$ we can associate an invertible sheaf $\mathcal{L}(D) \subseteq \mathcal{K}$ such that $\mathcal{L}(D)|_{U_i}$ is the free $\mathcal{O}_{U_i}$-submodule of $\mathcal{K}|_{U_i}$ generated by f_i^{-1}. The condition in the definition of Cartier divisor ensures that these patch together. One can prove that this procedure is reversible. ☐

Remark 2.43. Observe that if $\mathcal{K} \supseteq \mathcal{L}$ is invertible and $\mathcal{L} \cong \mathcal{O}_X$ (not just locally but globally!) then by taking the image $\mathcal{O}_X(X) \ni 1 \mapsto f \in \mathcal{L}(X)$ the Cartier divisor (X, f) is denoted by $\operatorname{div}(f)$, the divisor of f.

Corollary 2.44. *If D_1 and D_2 are Cartier divisors then $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ if and only if $D_1 - D_2 = \operatorname{div}(f)$ for some f.*

Definition 2.45. Recall that the *Picard group* of X is defined as

$$\operatorname{Pic}(X) := \text{Cartier divisors}/\{\operatorname{div}(f) \mid f \in K^\times\}$$

Proposition 2.46. *$D \mapsto \mathcal{L}(D)$ defines a group isomorphism*

$$\operatorname{Pic}(X) \to \{\textit{invertible sheaves on } X\}/\cong$$

where the group structure on the right hand side is induced by the tensor product as follows:

Definition 2.47. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent sheaves on X. We define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ as follows: For any affine open subset $\operatorname{Spec}(A) \cong U \subseteq X$ we have A-modules M and N such that

$$\mathcal{F}|_U \cong \tilde{M} \qquad \mathcal{G}|_U \cong \tilde{N}$$

hence we may define $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \widetilde{M \otimes_A N}$. Using the functoriality of tensor product, these patch together. In particular, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p \cong \mathcal{F}_p \otimes_{\mathcal{O}_X} \mathcal{G}_p$ as $\otimes_A$ commutes with colimits like taking stalks on the affine open $U = \operatorname{Spec}(A)$.

Lemma 2.48. *For $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes \mathcal{G}$ induces an abelian group structure on the invertible sheaves on X modulo isomorphism.*

Proof. First, we have to prove that if $\mathcal{F}$ and $\mathcal{G}$ are invertible then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is also invertible. This is true as for any affine open subset $U \cong \operatorname{Spec}(A) \subseteq X$ such that $\mathcal{F}|_U \cong \mathcal{G}|_U \cong \mathcal{O}_U$ i.e. as $\mathcal{O}_U$-modules, $\mathcal{F}|_U \cong \mathcal{G}|_U \cong \tilde{A}$. Therefore, $\mathcal{F} \otimes \mathcal{G}|_U \cong \widetilde{A \otimes_A A} \cong \tilde{A}$ where the isomorphism is given by the multiplication. So, invertible sheaves are really closed under tensor product. The associative and unital property comes from the similar properties of $\otimes_A$.

The inverse of $\mathcal{L}$ is defined as follows: $\mathcal{L}^\vee := \underline{\operatorname{Hom}}(\mathcal{L}, \mathcal{O}_X)$ i.e.

$$\mathcal{L}^\vee(U) := \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U)$$

also patch together to give an invertible sheaf because $\mathcal{L}|_U \cong \mathcal{O}_U$ implies $\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \cong \mathcal{O}_U$ as $\mathcal{O}_U$-modules. Moreover, $\mathcal{L}^\vee$ is an inverse for $\mathcal{L}$ since

$$(\mathcal{L} \otimes \mathcal{L}^\vee)(U) \to \mathcal{O}_X(U)$$

$$(s, \varphi) \mapsto \varphi(s)$$

This is an isomorphism as for U being sufficiently small, $\mathcal{L}|_U \cong \mathcal{O}_U$. □

Remark 2.49. Note that the problem at the inverse was to find a morphism that is well-defined, no matter which trivialization we choose. For the evaluation map, it is clear.

Remark 2.50. Recall that the Picard group of the projective space $\operatorname{Pic}(\mathbb{P}^n_A) \cong \mathbb{Z}$ for any ring A where any hyperplane is (the same) generator, as a Weyl divisor which is now the same as a Cartier divisor. What is the invertible sheaf corresponding to $[H]$?

Definition 2.51. (Serre) Define the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_A$ as follows: set

$$D_+(x_i) = \operatorname{Spec} A\big[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\big]$$

Then patch these together on $D_+(x_i) \cap D_+(x_j)$ using the identification $f \mapsto \frac{x_j}{x_i} f$. One can check that it defines a patching data.

Similarly, if we patch together using $f \mapsto \left(\frac{x_j}{x_i}\right)^n f$ for $n > 0$ we get $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$.

Take the hyperplane $H := V(x_0) \subseteq \mathbb{P}^n_A$ and consider $H \cap D_+(x_i) \subseteq D_+(x_i)$ is defined by $V\left(\frac{x_0}{x_i}\right)$. The Cartier divisor of H is $\left(D_+(x_i), \frac{x_0}{x_i}\right)$. Hence, $\mathcal{L}(D_H)$ is defined by $\frac{x_i}{x_0}$ on $D_+(x_i)$. On $D_+(x_i) \cap D_+(x_j)$ we have $\frac{x_j}{x_0} = \frac{x_j}{x_i} \cdot \frac{x_i}{x_0}$. Hence, we indeed get $\mathcal{O}(1)$. In fact, we get more $H = V(x_0)$ defines a global section "x_0" $\in \mathcal{O}(1)(\mathbb{P}^n)$ such that $x_0|_{D_+(x_i)} = \frac{x_0}{x_i}$. Similarly, $x_1, \dots, x_n$ define global sections in $\mathcal{O}(1)(\mathbb{P}^n)$ and, moreover, for all $p \in \mathbb{P}^n$ there exists an i such that $(x_i)_p$ generate $\mathcal{O}(1)_p$ as an $\mathcal{O}_{X,p}$-module.

Corollary 2.52. *There exists $n+1$ global sections in $x_0, \dots, x_n \in \mathcal{O}(1)(\mathbb{P}^n)$ such that for all $p \in \mathbb{P}^n$ there exists an i such that $(x_i)_p$ generate $\mathcal{O}(1)_p$ as an $\mathcal{O}_{X,p}$-module. This property is expressed as $\mathcal{O}(1)$ is* generated by global sections*.*

Definition 2.53. If $\varphi : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent sheaf on X then we define the *pullback sheaf* $\varphi^*\mathcal{F}$ as follows: If $\operatorname{Spec}(A) \cong U \subseteq Y$ is such that $\mathcal{F}|_U \cong \tilde{M}$ and $\operatorname{Spec}(B) \cong V \subseteq X$ is an affine open such that $\varphi(V) \subseteq U$ then $\varphi^*\mathcal{F}|_V := \widetilde{M \otimes_A B}$ where the morphism that makes B and A-module is the one that corresponds to the map $V \to U$.

Proposition 2.54. *If $\varphi : X \to \mathbb{P}^n$ is a morphism then $\varphi^*\mathcal{O}(1)$ is an invertible sheaf on X that is generated by global sections. This induces a bijection between morphisms $X \to \mathbb{P}^n_A$ of* $\operatorname{Spec}(A)$*-schemes and invertible sheaves $\mathcal{L}$ on X generated by $n+1$ global sections.*

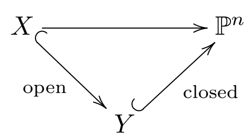
Remark 2.55. The argument is the same as in the case of base point free linear systems in the previous class.

Proof. Consider the open subset $X_i := \{p \in X \mid (s_i)_p \text{ generates } \mathcal{L}_p \text{ over } \mathcal{O}_{X,p}\} \subseteq X$. By assumption, $X = \cup X_i$. Then for all j there exists a unique $f_i \in \mathcal{O}_X(X_i)$ such that $s_j|_{X_i} = f_j s_i|_{X_i}$. Assume for simplicity that $X = \mathrm{Spec} B$ (otherwise cover by open affines). Then we may define a map $\mathrm{Spec} B \to D_+(x_i)$ as follows: we define the corresponding map

$$A\Big[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\Big] \to B \qquad \frac{x_j}{x_i} \mapsto f_j$$

□

Definition 2.56. Let $\mathcal{L}$ is a *very ample sheaf* on X if it is generated by global sections $x_0, \dots, x_n$ such that the induced map $X \to \mathbb{P}^n$ is an immersion, i.e. it factorizes as



and $\mathcal{L}$ is ample if there exists an n such that $\mathcal{L}^{\otimes n}$ is very ample.

Remark 2.57. Note that the open embedding part can sometimes be left out as if the morphism is finite then the image is always closed.

3 Cohomology

Definition 3.1. Let $\mathcal{C}$ be abelian category. $\mathcal{C}$ has *enough injectives* if every object $C \in \mathcal{C}$ can be embedded into an injective object. Then every $C \in \mathcal{C}$ has an *injective resolution*, i.e. an exact sequence

$$0 \to C \to I^0 \to I^1 \to \dots$$

where all I^i are exact. Given $F : \mathcal{C} \to \mathcal{D}$ a left exact functor into an abelian category $\mathcal{D}$, one defines the *i-th right derived functor as*

$$R^i F(C) := H^i\big(F(I^\bullet)\big)$$

Fact 3.2. *It does not depend on the injective resolution. The fact that F is left exact implies that $R^0 F = F$. Moreover, for any short exact sequence $0 \to A \to B \to C \to 0$ we can associate a long exact sequence of cohomologies:*

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$

We shall apply this in the following situation: let $\mathcal{C}_X$ be the category of sheaves of abelian groups on a topological space X.

Example 3.3.

1. Consider the *global sections functor* $F : \mathcal{F} \to \Gamma(X, \mathcal{F}) := \mathcal{F}(X)$. One can check that it is left exact.

2. Another example is the pushforward: let $\varphi : X \to Y$. Then $\mathcal{F} \to \varphi_* \mathcal{F}$ is left exact.

Lemma 3.4. *$\mathcal{C}_X$ has enough injectives.*

Proof. Take an $\mathcal{F} \in \mathcal{C}_X$ and for all $p \in X$ let $\mathcal{F}^p$ be the skyscraper sheaf defined by

$$\mathcal{F}^p(U) = \begin{cases} \mathcal{F}_p & p \in U \\ 0 & p \notin U \end{cases}$$

There exists a natural morphism of sheaves $\mathcal{F} \to \mathcal{F}^p$ taking either the stalk or zero. Then there is an injective morphism $\mathcal{F} \to \prod_{p\in X} \mathcal{F}^p$ (injectivity follows from the sheaf axioms). Now, for all $p \in X$, choose an embedding $\mathcal{F}_p \hookrightarrow I_p$ for some injective abelian group I_p. This induces an embedding

$$\mathcal{F} \hookrightarrow \prod_{p\in X} \mathcal{F}^p \hookrightarrow \prod_{p\in X} I^p$$

We claim that $\prod I^p$ is injective. For this, it is enough to prove I^p is injective for any $p \in X$ since product of injectives is injective. This follows from the injectivity of I_p as an abelian group since a morphism of sheaves $\mathcal{F} \to I^p$ always factors through $\mathcal{F}^p$ so we can check that the injectivity. ☐

Corollary 3.5. *We may define $H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$ and $(R^i f_*)\mathcal{F}$ for a morphism $f : X \to Y$.*

Definition 3.6. A sheaf $\mathcal{F} \in \mathcal{C}_X$ is flabby (or flasque in french) if for any $V \subseteq U$, $\mathcal{F}(U) \to \mathcal{F}(V)$ is onto.

Proposition 3.7. *If X is locally connected, then every injective sheaf is flabby.*

Remark 3.8. The converse is not true, for a counterexample take the skyscraper sheaf.

Proof. Observe that given an open subset $U \subseteq X$ there exists a unique sheaf $\mathbb{Z}_U$ on X such that for a connected open subset $V \subseteq X$

$$\mathbb{Z}_U(V) = \begin{cases} \mathbb{Z} & V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.9. *For any sheaf $\mathcal{F}$ on X, for any open subset $U \subseteq X$ we have $\mathrm{Hom}(\mathbb{Z}_U, \mathcal{F}) \xrightarrow{\cong} \mathcal{F}(U)$. In other words, $\mathbb{Z}_U$ represents the evaluation functor.*

Proof. For any morphism $\mathbb{Z}_U \xrightarrow{\varphi} \mathcal{F}$ we can associate $\varphi(1_U) \in \mathcal{F}(U)$ to it. Conversely, if $s \in \mathcal{F}(U)$ then there exists a unique $\varphi : \mathbb{Z}_U \to \mathcal{F}$, $\mathbb{Z}_U(V) \ni 1 \mapsto s|_V$ for any connected open subset $V \subseteq U$. These are inverses to each other. ☐

Now, to prove the proposition, take an open inclusion $V \subseteq U$. We need to prove that $\mathcal{F}(U) \to \mathcal{F}(V)$ is onto. Note that the morphism is $\mathcal{F}(U) \cong \mathrm{Hom}(\mathbb{Z}_U, \mathcal{F}) \to \mathrm{Hom}(\mathbb{Z}_V, \mathcal{F}) \cong \mathcal{F}(V)$ where the morphism is the one induced by $\mathbb{Z}_V \hookrightarrow \mathbb{Z}_U$. Hence, by the injectivity of $\mathcal{F}$ we get that $\mathrm{Hom}(\mathbb{Z}_U, \mathcal{F}) \to \mathrm{Hom}(\mathbb{Z}_V, \mathcal{F})$ is surjective. ☐

Proposition 3.10. *If $\mathcal{F}$ is flabby then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Such an object is called acyclic.*

Lemma 3.11.

1. *If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact and $\mathcal{F}$ is flabby then $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$ is exact.*
2. *If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact and $\mathcal{F}$ and $\mathcal{G}$ are flabby then $\mathcal{H}$ is also flabby.*

Proof. For the proof of *a*), see the notes on the website of the lecturer. For the proof of *b*), take an open inclusion $V \subseteq U$. Then

$$\begin{array}{ccc} \mathcal{G}(U) & \xrightarrow{\text{as } \mathcal{G} \text{is flabby}} & \mathcal{G}(V) \\ \downarrow & & \downarrow {\scriptstyle \text{by } a)} \\ \mathcal{H}(U) & \longrightarrow & \mathcal{H}(V) \end{array}$$

hence $\mathcal{H}(U) \to \mathcal{H}(V)$ must be onto too. □

Proof of Proposition 3.10. Choose an embedding $\mathcal{F} \to \mathcal{I}$ where $\mathcal{I}$ is injective. Then, we have an exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ where $\mathcal{F}$ and $\mathcal{I}$ are flabby (by Proposition 3.7). Hence, by part *b*) of the lemma, $\mathcal{G}$ is also flabby. Therefore, we get a long exact sequence of cohomologies

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{I}) = 0$$

using that $\mathcal{I}$ is injective. However, by part *a*) of the lemma, $\mathcal{I} \to \mathcal{G}$ is surjective, hence $H^1(X, \mathcal{F}) = 0$. Then, by induction one can prove that all the cohomologies are zero (by dimension shift). □

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Remark 3.12. Note that we used the cohomology construction for two functors: for the global sections functors $\Gamma(X, .)$ and for the pushforward f_*. It is important to keep in mind that, by definition the cohomologies of the first are abelian groups while the cohomologies of the latter are also sheaves themselves.

3.1 Serre's vanishing theorem

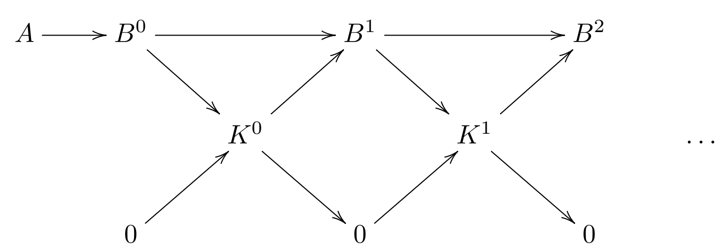
Lemma 3.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories such that $\mathcal{C}$ has enough injectives and let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor. Assume that $A \in \mathcal{C}$ such that there exists a resolution (i.e. an exact sequence)*

$$0 \to A \to B^0 \to B^1 \to B^2 \to \dots$$

such that $R^iF(B^j) = 0$ for all $j \geq 0$ and $i > 0$. Then $R^iF(A) = H^iF(B^\bullet)$. In short, cohomologies can be computed on acyclic objects.

Corollary 3.14. *$H^i(X, \mathcal{F})$ may be computed using flabby resolutions. (Apply Lemma 3.13 and Proposition 3.10 saying that flabby is acyclic.)*

Proof of Lemma 3.13. Split the resolution into short exact sequences



Then take the corresponding long exact sequences, e.g.

$$0 \to F(A) \to F(B^0) \to F(K^0) \to R^1F(A) \to 0$$

using $R^1F(B^0) = 0$. Hence,

$$R^1F(A) \cong \operatorname{Coker}\big(F(B^0) \to F(K^0)\big) = \operatorname{Coker}\Big(F(B^0) \to \operatorname{Ker}\big(F(B^1) \to F(B^2)\big)\Big) = H^1F(B^\bullet)$$

our statement for $i = 1$.

Moreover, for the other short exact sequences of the form

$$0 \to K^{i-1} \to B^i \to K^i \to 0$$

and assuming that $R^jF(B^i) = 0$ for all $j > 0$, we get $R^jF(K^i) \cong R^{j+1}F(K^{i-1})$ for all i. Putting these together for fix j and all i, we get $R^jF(K^0) \cong R^{j+1}F(A)$.

Now, putting everything together, we can write

$$R^{j+1}F(A) \cong R^jF(K^0) \cong R^{j-1}F(K^1) \cong \ldots \cong R^1F(K^{j-1}) \cong \mathrm{Coker}\Big(F(B^j) \to \mathrm{Ker}\big(F(B^{j+1}) \to F(B^{j+1})\big)\Big)$$

where the right hand side is just $H^{j+1}(F(B^\bullet))$. □

Application: If $f : X \to Y$ is a morphism of topological spaces and $\mathcal{F}$ is a sheaf on X then $R^if_*\mathcal{F}$ is the sheafification of the presheaf $V \mapsto H^i\big(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}\big)$. In particular, the derived functor of the pushforward can be expressed using sheaf cohomology.

Proof. Take an injective resolution $\mathcal{F} \to I^\bullet$. Then $R^if_*\mathcal{F} = H^i\big(f_*(I^\bullet)\big)$, by definition. The right hand side is the sheaf associated with the presheaf

$$V \mapsto H^i\big(\Gamma(V, f_*I^\bullet|_V)\big)$$

by the definition of sheaf cokernel. However, this definition is the same as $H^i\big(\Gamma(f^{-1}(V), I^\bullet|_{f^{-1}(V)}\big)$ by the definition of the pushforward. Now, we can use that I^j is injective for all j, in particular, it is flabby (Proposition 3.10). Fortunately, a restriction $I^j|_{f^{-1}(V)}$ of the flabby sheaf I^j is also flabby (by the definition). So $\mathcal{F}|_{f^{-1}(V)} \to I^\bullet|_{f^{-1}(V)}$ is a flabby resolution. By Corollary 3.14, $H^i\big(\Gamma(f^{-1}(V), I^\bullet|_{f^{-1}(V)})\big) = H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. □

Theorem 3.15. (Serre's vanishing theorem) *Let X be an affine scheme, $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Recall: A morphism of schemes $\varphi : X \to Y$ is affine if for all $V \subseteq Y$ affine open, $\varphi^{-1}(V)$ is affine open.

Corollary 3.16. *If $\varphi : X \to Y$ is an affine morphism, $\mathcal{F}$ a quasi-coherent sheaf on X then $R^i\varphi_*\mathcal{F} = 0$ for all $i > 0$.*

Proof. By the Application of Corollary 3.14, it is enough to prove that $H^i\big(\varphi^{-1}(V), \mathcal{F}|_{\varphi^{-1}(V)}\big) = 0$ for all $V \subseteq Y$ affine open. But here, one can apply Theorem 3.15 for the affine scheme $\varphi^{-1}(V)$. □

Corollary 3.17. *For $\varphi : X \to Y$ affine, $\mathcal{F}$ quasi-coherent on X, we have*

$$H^i(Y, \varphi_*\mathcal{F}) \cong H^i(X, \mathcal{F})$$

Remark 3.18. Note that it is usually rare, but quasi-coherent sheaves are nice enough.

Proof. Take an injective resolution $\mathcal{F} \to I^\bullet$ and apply φ_* giving $\varphi_*\mathcal{F} \to \varphi_*I^\bullet$. As I^j is flabby for all j, φ_*I^j is flabby too (it can be checked by the definition of flabbiness). Moreover, $\varphi_*\mathcal{F} \to \varphi_*I^\bullet$ is a flabby resolution because $R^i\varphi_*\mathcal{F} = H^i(\varphi_*I^\bullet) = 0$, by Corollary 3.16. So, by Corollary 3.14, we may compute the cohomologies of $\varphi_*\mathcal{F}$ using $\varphi_*I^\bullet$. Therefore,

$$H^i(Y, \varphi_*\mathcal{F}) = H^i\big(\Gamma(Y, \varphi_*I^\bullet)\big) = H^i\big(\Gamma(X, I^\bullet)\big) = H^i(X, \mathcal{F})$$

as we stated. (The implication $\Gamma(Y, \varphi_*I^\bullet) = \Gamma(X, I^\bullet)$ in the middle equality follows by the definitino of pushforward.) □

Proof of Serre's vanishing theorem. **Step1:** We show that given $s \in H^1(X, \mathcal{F})$, there exists an affine open covering $U_1, \ldots, U_r$ of X (finite since an affine scheme is (quasi-)compact), such that the image of s in $H^1(X, \mathcal{F}_j)$ is zero for all $j = 1, \ldots, r$. (Recall that $\mathcal{F}_j$ is the quasi-coherent sheaf on X defined by $\mathcal{F}_j(U) := \mathcal{F}(U \cap U_j)$.) Note that the covering may depend on the section!

Proof of Step 1. Embed $\mathcal{F}$ in an injective sheaf $\mathcal{I}$: $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{C} \to 0$. Note that $\mathcal{I}$ is not necessarily quasi-coherent. Now, take the global sections giving

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{C}(X) \to H^1(X, \mathcal{F}) \to 0$$

Any element $s \in H^1(X, \mathcal{F})$ has a preimage $t \in \mathcal{C}(X)$. By $\mathcal{I} \to \mathcal{C} \to 0$, there exists an open covering $U_1, \ldots, U_r$ such that $t|_{U_j}$ lifts to $t_j \in \mathcal{I}(U_j)$. We may assume that all the U_j's are affine as affine subsets form a basis.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_j & \longrightarrow & \mathcal{I}_j & \longrightarrow & \overline{\mathcal{C}_j} & \longrightarrow & 0
\end{array}$$

where $\mathcal{I}_j(U) = \mathcal{I}(U \cap U_j)$ and $\overline{\mathcal{C}_j} = \mathrm{Coker}(\mathcal{F}_j \to \mathcal{I}_j)$. (It is true that $\overline{\mathcal{C}_j} = \mathcal{C}_j$ but we don't know that yet.) We also have the corresponding commuatative diagram of global sections

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \mathcal{C}(X) & \longrightarrow & H^1(X, \mathcal{F}) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathcal{F}_j(X) & \longrightarrow & \mathcal{I}_j(X) & \longrightarrow & \overline{\mathcal{C}_j}(X) & \longrightarrow & H^1(X, \mathcal{F}_j)
\end{array}$$

as the long exact sequence of cohomologies is functorial. The previously constructed $t_j \in \mathcal{I}_j(X)$ hence its image in $H^1(X, \mathcal{F}_j)$ is zero but that is the same as the image of $s \in H^1(X, \mathcal{F})$ as both are images of $t \in \mathcal{C}(X)$. Note that we have not used neither the quasi-coherent property of the sheaf and nor the affine hypothesis. □

Step2: We show that $H^1(X, \mathcal{F}) = 0$.

Proof of Step 2. Let $s \in H^1(X, \mathcal{F})$. Choose $U_1, \ldots, U_r$ as in step 1. Then $H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}_j)$ kills s for all $j = 1, \ldots r$. Also, have a natural injective morphism of sheaves

$$\begin{array}{rcl}
\mathcal{F} & \to & \prod_{j=1}^{r} \mathcal{F}_j \\
\mathcal{F}(U) \ni t & \mapsto & t|_{U \cap U_j}
\end{array}$$

Then we can form the cokernel $\mathcal{G} := \left(\mathcal{F} \to \prod \mathcal{F}_j\right)$. As all the $\mathcal{F}_j$'s are quasi-coherent, $\mathcal{G}$ is also quasi-coherent since the ˜ functor is exact.

Since X is affine and everything is quasi-coherent (as it was not when we only had $\mathcal{I}$) we have the exact sequence

$$0 \to \mathcal{F}(X) \to \prod \mathcal{F}_j(X) \to \mathcal{G}(X) \to 0$$

However, we have another long exact sequence

$$0 \to \mathcal{F}(X) \to \prod \mathcal{F}_j(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{F}) \hookrightarrow \prod H^1(X, \mathcal{F}_j)$$

where the term on the right hand side is zero as we proved in step 1. Hence $H^1(X, \mathcal{F}) = 0$.

The general case for higher cohomologies follows by a not totally straightforward dimensionshift: see the online lecture notes. Nontriviality lies in the fact the covering is section-dependent. □

□

3.2 Cech cohomology

Theorem 3.19. (Grothendieck) *Assume that X is a topological space of Krull dimension at most n (i.e. every descending chain of irreducible closed subsets has lenght at most n). Let $\mathcal{F}$ be a sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > n$.*

Remark 3.20. Note that here we do not even assume that X is a scheme.

Theorem 3.21. *Let A be a ring and $\mathcal{F}$ a quasi-coherent sheaf on $\mathbb{P}^n_A$. Then $H^i(\mathbb{P}^n_A, \mathcal{F}) = 0$ for $i > n$.*

Remark 3.22. It follows from the previous theorem if A is a field, but generally it does not.

The latter theorem follows by the following:

Theorem 3.23. (Serre) *If X is a separated scheme that can be covered by $n+1$ affine open subschemes and $\mathcal{F}$ is a quasi-coherent sheaf on X then $H^i(X, \mathcal{F}) = 0$ for $i > n$.*

Definition 3.24. Let X be a topological space and $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of X where I is well-ordered. Given a finite subset $\{i_0, \ldots, i_p\} \subseteq I$, we define

$$U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$$

Suppose that $\mathcal{F}$ is a sheaf on X. We define the *Cech complex* of $\mathcal{U}$ on $\mathcal{F}$ is the complex of abelian groups $C^\bullet(\mathcal{U}, \mathcal{F})$ where

$$C^\bullet(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

Together with the coboundary

$$\begin{aligned} \mathrm{d}^p : C^p(\mathcal{U}, \mathcal{F}) \quad &\to \quad C^{p+1}(\mathcal{U}, \mathcal{F}) \\ (\alpha_{i_0,\ldots,i_p})_{i_0,\ldots,i_p} \quad &\mapsto \quad \big((\mathrm{d}^p\alpha)_{i_0,\ldots,i_{p+1}}\big)_{i_0,\ldots,i_{p+1}} := \Big(\sum_{k=0}^{p+1}(-1)^k(\alpha_{i_0,\ldots,\hat{i_k},\ldots,i_{p+1}})|_{U_{i_0,\ldots,i_{p+1}}}\Big)_{i_0,\ldots,i_{p+1}} \end{aligned}$$

One can check that $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$ so it is really a complex.

Remark 3.25. Note that here, we are working over a "naked" topological space that is not necessarily a scheme.

Lemma 3.26. *Assume that $\mathcal{U}$ contains $U_i = X$. Then the complex*

$$0 \to \mathcal{F}(X) \xrightarrow{\varepsilon} C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \ldots$$

is exact, where ε is induced by the restrictions $s \mapsto (s|_{U_i})$.

Proof. We may assume that $1 \in I$ and $U_1 = X$. Exactness at $C^0(\mathcal{U}, \mathcal{F})$ follows from the sheaf axioms. To get exactness at $C^i(\mathcal{U}, \mathcal{F})$ for $i > 0$ we prove that $\mathrm{id}_{C^\bullet(\mathcal{U},\mathcal{F})}$ is homotopic to zero. This means that for all $p > 0$ we define a map $k^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ such that

$$\mathrm{d}^{p-1} \circ k^p + k^{p+1} \circ \mathrm{d}^p = \mathrm{id}_{C^p(\mathcal{U},\mathcal{F})}$$

This condition implies that $H^p\big(C^\bullet(\mathcal{U}, \mathcal{F})\big) = 0$ (as usual, "homotopic implies homologic"). So let

$$k^p(\alpha_{i_0,\ldots,i_p}) := \begin{cases} 0 & \text{if } i_0 \neq 1 \\ \alpha_{i_0,\ldots,i_p} & \text{if } i_0 = 1 \end{cases}$$

which makes sense since $U_1 = X$ hence $\alpha_{i_0,\ldots,i_p}$ can be interpreted as a section on an intersection of p subsets instead of $p+1$. ∎

Definition 3.27. Sheafification of $C^p(\mathcal{U},\mathcal{F})$: We define

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) := \prod_{i_0<\dots<i_p} j_*\mathcal{F}|_{U_{i_0,\dots,i_p}}$$

where $j : U_{i_0,\dots,i_p} \hookrightarrow X$ is the inclusion. Moreover, we define the coboundary $\mathrm{d}^p : \mathcal{C}^p(\mathcal{U},\mathcal{F}) \to \mathcal{C}^p(\mathcal{U},\mathcal{F})$ the same way as before. This way, we get a complex of sheaves $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ with the property

$$\Gamma\big(X,\mathcal{C}^p(\mathcal{U},\mathcal{F})\big) = C^p(\mathcal{U},\mathcal{F})$$

for all $p \geq 0$.

Proposition 3.28. *The sequences of sheaves* $0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U},\mathcal{F}) \to \mathcal{C}^1(\mathcal{U},\mathcal{F}) \to \dots$ *is exact.*

Remark 3.29. Note that we do not assume $U_1 = X$ but that is not contradictory as it is an exact sequence of sheaves and not of presheaves, i.e. the global sections may not form an exact sequence.

Proof. It is enough to check exactness on stalks. In fact, it is enough to check that for sufficiently small open neighborhoods $V \subseteq X$, $0 \to \mathcal{F}(V) \to \mathcal{C}^0(\mathcal{U},\mathcal{F})(V) \to \mathcal{C}^1(\mathcal{U},\mathcal{F})(V) \to \dots$ is exact. Say V is sufficiently small if there exists a $U_i \subseteq U$ such that $V \subseteq U_i$. Then we may apply the previous Lemma 3.26. □

Now, we get back to scheme theory from generalities.

Lemma 3.30. *If* $\mathcal{U} = \{U_1,\dots,U_r\}$ *is a finite affine open covering of a separated scheme* X *and* $\mathcal{F}$ *is a quasi-coherent sheaf on* X *then* $H^i\big(X,\mathcal{C}^p(\mathcal{U},\mathcal{F})\big) = 0$ *for all* $p \geq 0$ *and* $i > 0$.

Proof. Recall that

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\dots<i_p} \mathcal{F}_{i_0,\dots,i_p} = \prod_{i_0<\dots<i_p} j_*\big(\mathcal{F}|_{i_0,\dots,i_p}\big)$$

hence $H^i\big(X,\mathcal{C}^p(\mathcal{U},\mathcal{F})\big) = \prod_{i_0<\dots<i_p} H^i(X,\mathcal{F}_{i_0,\dots,i_p})$ as cohomology commutes with finite products. By separatedness of X, $U_{i_0,\dots,i_p}$ is affine (see Proposition 1.52). Hence,

$$H^i\big((X,j_*(\mathcal{F}|_{U_{i_0,\dots,i_p}})\big) = H^i(U_{i_0,\dots,i_p},\mathcal{F}|_{U_{i_0,\dots,i_p}})$$

as we can apply Corollary 3.17 for the affine morphism j (as $U_{i_0,\dots,i_p}$ is affine). However, $\mathcal{F}|_{U_{i_0,\dots,i_p}}$ is quasi-coherent as $\mathcal{F}$ itself was quasi-coherent, so by Serre's vanishing Theorem 3.15, we get the claim. □

Proof of Theorem 3.23. By Proposition 3.28 the Cech complex is a resolution, and by Lemma 3.30 it is an acyclic resolution. Therefore, by the formal Lemma 3.13, we may compute the cohomology of $\mathcal{F}$ using $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$. However, $\mathcal{C}^p(\mathcal{U},\mathcal{F}) = 0$ if $p > n$, by definition (the intersecting open subsets must be distinct) so we get the statement of the theorem. □

Now, we define Cech cohomology. In the theorem, we already used it, i.e. we have shown that for a finite covering with affine opens, Cech cohomology is the same as sheaf cohomology.

Definition 3.31. Let $\mathcal{F}$ be a sheaf on a scheme X and $\mathcal{U}$ an open covering. The Cech complex $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ gives rise to the cohomology $H^p(\mathcal{U},\mathcal{F}) = H^p\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$. Introduce the partial order on all coverings: we say that $\mathcal{U}$ is coarser than $\mathcal{V}$ (notation: $\mathcal{U} \prec \mathcal{V}$) if for any $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subseteq U$. In this case, one can check (!) that there exists a map $H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{V},\mathcal{F})$. Then, one defines

$$\check{H}^p(X,\mathcal{F}) := \lim_{\mathcal{U}\to} H^p(\mathcal{U},\mathcal{F})$$

for all $p \geq 0$.

Theorem 3.32. *If* X *is a separated scheme and* $\mathcal{F}$ *is a quasi-coherent sheaf then* $\check{H}^i(X,\mathcal{F}) \cong H^i(X,\mathcal{F})$.

Seventh Lecture, 3rd of March

3.3 Serre's projective morphism theorem

Definition 3.33. Let A be a ring. Then a *projective morphism* of X to $\operatorname{Spec}(A)$ is a closed immersion $X \hookrightarrow \mathbb{P}^n_A$ over $\operatorname{Spec}(A)$ i.e. a commutative diagram of the form

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathbb{P}^n_A \\ & \searrow & \downarrow \\ & & \operatorname{Spec}(A) \end{array}$$

Observation: If $\mathcal{F}$ is a quasi-coherent sheaf on X and we have a fixed projective morphism $X \to \operatorname{Spec}(A)$ then for all $i \geq 0$, $H^i(X, \mathcal{F})$ is an A-module.

Proof. One shows that $\mathcal{F}$ has a resolution by injective $\mathcal{O}_X$-modules (this can be proved the same way as for general sheaves). These are also flabby (the proof is analogous to Proposition 3.7). Hence, we may compute $H^i(X, \mathcal{F})$ on this resolution of injective $\mathcal{O}_X$-modules. This way, we obtain the complex $\Gamma(X, I^\bullet)$ that are modules over $\Gamma(X, \mathcal{O}_X)$. The latter is A as X is projective over $\operatorname{Spec}(A)$ and this morphism is proper as every projective morphism is proper (even in this generality), giving that $\Gamma(X, \mathcal{O}_X) = A$.

On $\mathbb{P}^n_A$ we have $\mathcal{O}(m)$ for $m \in \mathbb{Z}$ (obtained by patching together $\mathcal{O}_{\mathbb{P}^n}|_{D_+(x_i)}$ via $f \mapsto \left(\frac{x_j}{x_i}\right)^m f$). We can pull it back along $i : X \to \mathbb{P}^n_A$ giving $i^*\mathcal{O}(m) =: \mathcal{O}_X(m)$ a quasi-coherent sheaf on X. Then we may define the twist of $\mathcal{F}$, if it iss a quasi-coherent sheaf on X: let $\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}_X(m)} \mathcal{O}_X(m)$. □

Theorem 3.34. (Serre) *Assume that A is Noetherian, $\mathcal{F}$ is a coherent sheaf on X. Then*

1. $H^i(X, \mathcal{F})$ is a finitely generated A-module for all $i \geq 0$.

2. For m sufficiently large, $H^i\big(X, \mathcal{F}(m)\big) = 0$ for all $i > 0$.

Remark 3.35. In part b), m is independent of i. But in fact, the two possible meaning of part b) are the same by Serre's theorem, 3.23.

Remark 3.36. Grothendieck generalized this theorem to the case when $X \to \operatorname{Spec}(A)$ is just proper. Originally, Grothendieck proved this by reduction to the projective space but there is a proof that does not go by reduction, involving hard machinery.

Steps of the proof of Theorem 3.34:

1. We reduce to $X = \mathbb{P}^n_A$,
2. We reduce to the case of $\mathcal{F} = \mathcal{O}(m)$,
3. We solve the last case $X = \mathbb{P}^n_A$, $\mathcal{F} = \mathcal{O}(m)$.

Proof of Step 1. First, we need a so called *projection formula:*

Proposition 3.37. *Let $i : X \to Y$ be an affine morphism, $\mathcal{F}$ a quasi-coherent sheaf on X and $\mathcal{G}$ is a quasi-coherent sheaf on $\mathcal{G}$.*

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} i^*\mathcal{G}) \cong i_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$$

Proof. We do it for X and Y being affine then we patch together the general case. Let$X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$. Then i corresponds to an algebra map $A \to B$, while $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} = \tilde{N}$. The pullback and pushout is, by definition $i_*\mathcal{F} = M$ as an A-module via ρ, and $i^*\mathcal{G} = N \otimes_A B$. Then

$$i_*(\mathcal{F} \otimes i^*\mathcal{G}) = M \otimes_B \widetilde{(N \otimes_A} B) \cong \widetilde{M \otimes_A} N$$

as an A-module, which is the same as $i_*\mathcal{F} \otimes \mathcal{G} = M \mathbin{\tilde{\otimes}_A} N$. □

Now, we apply this proposition on $X \hookrightarrow \mathbb{P}^n_A =: Y$ where $\mathcal{F}$ is a quasi-coherent sheaf on X and $\mathcal{G} = \mathcal{O}(m)$. Then, we get

$$i_*\big(\mathcal{F} \otimes i^*\mathcal{O}(m)\big) = (i_*\mathcal{F}) \otimes \mathcal{O}(m)$$

In short, we get $i_*(\mathcal{F}(m)) \cong (i_*\mathcal{F})(m)$.

Therefore, we may compute

$$H^j(X, \mathcal{F}(m)) = H^j\big(X, (i_*\mathcal{F})(m))\big) \cong H^j\big(X, (i_*\mathcal{F})(m)\big)$$

where the first equality is the consequence of Serre's vanishing Theorem, 3.15 and the second equality follows by the above computations. $(j \geq 0)$ □

Proof of Step 2. We need the following key lemma:

Lemma 3.38. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_A$ then there exists a surjection $\mathcal{O}(m)^{\oplus r} \twoheadrightarrow \mathcal{F}$ for some $m \in \mathbb{Z}$ and $r \geq 0$.*

Proof. We look for a surjection $\mathcal{O}^r \twoheadrightarrow \mathcal{F}(m)$ for some $\in \mathbb{Z}$. (Indeed, we can tensor-multiply such a morphism via $\mathcal{O}(-m)$.) Hence, we look for global sections $s_1, \dots, s_r \in \Gamma(\mathbb{P}^n_A, \mathcal{F}(m))$ such that $(s_i)_p$ generate the stalk $\mathcal{F}(m)_p$ over $\mathcal{O}_{\mathbb{P}^n,p}$. (Indeed, if we have such $s_1, \dots, s_n$ then we can define the surjection $\mathcal{O}^r_{\mathbb{P}^n} \to \mathcal{F}(m)$ by

$$\mathcal{O}^r_{\mathbb{P}^n}(U) \ni (f_1, \dots, f_r) \mapsto \sum_{i=1}^r f_i s_i|_U$$

which is a surjection as it is a surjection on the stalks. Observe that for $\mathcal{F}|_{D_+(x_i)}$ we have such an $s_i|_{D_+(x_i)}$'s as $D_+(x_i)$ is affine hence a coherent sheaf corresponds to a finitely generated module that is clearly a quotient of a free of finite rank. Using this, it remains to prove that □

Claim 3.39. Given $t_i \in \mathcal{F}\big(D_+(x_i)\big)$ for some i, there exists a $\tilde{t} \in \Gamma\big(\mathbb{P}^n_A, \mathcal{F}(m)\big)$ for m sufficiently large such that $\tilde{t}|_{D_+(x_i)} = t_i$. (For the proof, see lecture notes.)

Then to prove Step 2 (assume Step 3!), by the lemma, we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}(m)^{\oplus r} \to \mathcal{F} \to 0$$

where $\mathcal{K}$ is a coherent sheaf by Noetherian property of $\mathbb{P}^n_A$ which follows by the Noetherin property of A. So we may take its long exact sequence of cohomologies

$$\dots \to H^i\big(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus r}\big) \to H^i\big(\mathbb{P}^n_A, \mathcal{F}\big) \to H^{i+1}\big(\mathbb{P}^n_A, \mathcal{K}\big) \to \dots$$

Here, the idea is to prove by induction. However, the theorem is nontrivial even for $i = 0$. So, instead, we start at high i: By Theorem 3.23, $H^{i+1}(\mathbb{P}^n_A, \mathcal{K}) = 0$ for $i > n-1$ as $\mathcal{K}$ is quasi-coherent. So use descending induction: the claim is true for $H^{i+1}(\mathbb{P}^n_A, \mathcal{K})$ by induction. Moreover, it is true for $H^i\big(\mathbb{P}^n_A, \mathcal{O}(m)\big)$ by Step 3. So we get that it is also true for $H^i\big(\mathbb{P}^n_A, \mathcal{F}\big)$ since H^i is additive, so $H^i\big(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus r}\big) \cong \oplus_{j=1}^r H^i\big(\mathbb{P}^n_A, \mathcal{O}(m)\big)$.

Remark 3.40. Note that even for the case $i = 0$, this is the way to prove.

□

Proof of Step 3. It is the following ("computational") theorem

Theorem 3.41. *For $n \geq 1$, $m \in \mathbb{Z}$ we have*

$$H^i\big(\mathbb{P}^n_A, \mathcal{O}(m)\big) = \begin{cases} A[x_0, \dots, x_n] & i = 0 \\ 0 & 0 < i < n \text{ or } i > n \\ S_m & i > n \end{cases}$$

where S_m is the A-submodule of the degree m part of $A[x_1^{-1}, \dots, x_n^{-1}]$ generated by monomials of the form $x_0^{\alpha_0} \dots x_n^{\alpha_n}$ such that $\alpha_j < 0$ for all j.

Note that this implies Serre's theorem in this special case of $\mathcal{O}(m)$.

Remark 3.42. In the book of Hartshorne (as in the original proof of Serre), it is proven by Cech cohomology, but here we follow a different route:

Lemma 3.43. *Consider the defining quotient morphism* $\pi : \mathbb{A}^{n+1}_A \backslash \{0\} \to \mathbb{P}^n_A$*. Then the pushforward is*

$$\pi_* \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}} \cong \oplus_{m\in\mathbb{Z}} \mathcal{O}(m)$$

Proof. Precisely, the map $\pi|_{D(x_i)} D(x_i) \to D_+(x_i) \subseteq \mathbb{P}^n$ is defined by the corresponding algebra homomorphism

$$A\big[x_0, \dots, x_n\big]_{x_i} \leftarrow A\big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\big]$$

Then

$$\pi_* \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}|_{D_+(x_i)} = A[x_0, \widetilde{\dots}, x_n]_{x_i}$$

where $A[x_0, \dots, x_n]_{x_i}$ is understood as an $A\big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\big]$-module. We have a decomposition

$$A[x_0, \dots, x_n]_{x_i} = \bigoplus_{m\in\mathbb{Z}} A\big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\big] x_i^m$$

for all $0 \le i \le n$. Moreover, the transition functions of $\pi_* \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}$ between

$$\bigoplus_{m\in\mathbb{Z}} A\big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\big] x_i^m \qquad \text{and} \qquad \bigoplus_{m\in\mathbb{Z}} A\big[\frac{x_0}{x_j}, \dots, \frac{x_0}{x_j}\big] x_j^m$$

given by $\oplus_m \Big(\frac{x_j}{x_i}\Big)^m$. This is the same as the transition functions of $\mathcal{O}(m)$ so we got the lemma. □

By Serre's vanishing theorem 3.15 we get that

$$H^i\big(\mathbb{A}^{n+1}\backslash\{0\}, \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}\big) \cong H^i\big(\mathbb{P}^n_A, \pi_* \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}\big) \cong$$

and by the previous lemma, it is

$$\cong H^i\big(\mathbb{P}^n_A, \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)\big) \cong \bigoplus_{m\in\mathbb{Z}} H^i\big(\mathbb{P}^n_A, \mathcal{O}(m)\big)$$

where we used that $\bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)$ is flabby as the infinite direct sum of flabby sheaves is flabby (a property that is not necessarily true for injectives). So it is enough to compute $H^i\big(\mathbb{A}^{n+1}\backslash\{0\}, \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}\big)$.

Proposition 3.44. *We have*

$$H^i\big(\mathbb{A}^{n+1}\backslash\{0\}, \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}\big) = \begin{cases} 0 & \text{if } i \neq 0, n \\ A[x_0, x_0^{-1}] & \text{if } i = 0 \text{ and } n = 0 \\ A[x_0, \dots, x_n] & \text{if } i = 0 \text{ and } n > 0 \\ S_m & \text{if } i = n \end{cases}$$

where S_m *is defined in the theorem.*

Clearly, the statement implies the theorem. Note, however that $n = 0$ will not play a role there, but we need here for the induction.

Proof. We use induction on n and i. For n arbitrary and $i = 0$ we already computed the result in an elementary way, in a previous semester. For $n = 0$ and $i > 0$ we have $H^i\big(\mathbb{A}^1\backslash\{0\}, \mathcal{O}_{\mathbb{A}^1\backslash\{0\}}\big) = 0$ by Serre's vanishing theorem 3.15. Now, let $n > 0$ and $i > 1$. Consider $D(x_n) \overset{j}{\hookrightarrow} \mathbb{A}^{n+1}\backslash\{0\}$. Then, one can check that we have a short exact sequences

$$0 \to \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}} \to j_*\mathcal{O}_{D(x_n)} \to \bigoplus_{l=1}^{\infty} \mathcal{O}_{\mathbb{A}^n\backslash\{0\}} \cdot x_n^{-l} \to 0$$

as $\mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}$ is a sub and the elements of $j_*\mathcal{O}_{D(x_n)}$ modulo $\mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}$ are multiples of x_n^{-l} for some l.

Hence, $H^i\big(\mathbb{A}^{n+1}\backslash\{0\}, j_*\mathcal{O}_{D(x_n)}\big) = H^i\big(D(x_n), \mathcal{O}_{D(x_n)}\big) = 0$ for $i > 0$ by Theorem 3.17. Therefore, we get

$$H^1\big(\mathbb{A}^{n+1}\backslash\{0\}, \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}\big) = \mathrm{Coker}\Big((H^0\Big(D(x_n), \mathcal{O}_{D(x_n)} \to \bigoplus_{l=1}^{\infty} \mathcal{O}_{\mathbb{A}^n\backslash\{0\}} \cdot x_n^{-l}\Big)\Big)$$

and

$$H^i\big(\mathbb{A}^{n+1}\backslash\{0\}, \mathcal{O}_{\mathbb{A}^{n+1}\backslash\{0\}}\big) \cong \bigoplus_{l=1}^{\infty} H^{i-1}\big(\mathbb{A}^n\backslash\{0\}, \mathcal{O}_{\mathbb{A}^n\backslash\{0\}}\big) \cdot x_n^{-l}$$

for $i > 1$. The latter equality gives the possibility to use induction.

If $i = 1$, $n = 1$ then one can check that we can explicitly compute the above mentioned cohomologies,

$$\mathrm{Coker}\Big(A[x_0, x_1, x_1^{-1}] \to \bigoplus_{l=1}^{\infty} A[x_0, x_0^{-.1}]x_1^{-l}\Big) \cong \bigoplus_{l=1}^{\infty}\bigoplus_{m=1}^{\infty} A \cdot x_0^{-m}x_1^{-l}$$

While, if $i = 1$, $n > 1$ then these cohomologies are

$$\mathrm{Coker}\Big(A[x_0, x_1, \ldots, x_n, x_n^{-1}] \to \bigoplus_{l=1}^{\infty} A[x_0, \ldots, x_{n-1}]x_n^{-l}\Big) = 0$$

by induction. □

□

Corollary 3.45. *If X is projective over* $\mathrm{Spec}\Bbbk$ *and $\mathcal{F}$ is a coherent sheaf then $H^i(X, \mathcal{F})$ are finite dimensional $\Bbbk$-vector spaces that are zero for i sufficiently large. In fact, $H^i(X, \mathcal{F}) = 0$ if $i > n$, but even more is true: it is zero if $i > \dim X$ as X has a finite (in particular, it is affine) surjection on $\mathbb{P}^{\dim X}$ so it can be covered by $d + 1$ affine opens.)*

3.4 Riemann-Roch theorem

Definition 3.46. Define the Euler-(Poincaré) characteristic as

$$\chi(\mathcal{F}) = \sum_i (-1)^i \mathrm{dim}_{\Bbbk} H^i(X, \mathcal{F})$$

Problem 3.47 (Riemann-Roch problem:)**.** Compute $\chi(\mathcal{F})$.

Setup: Assume that X is integral closed subscheme of $\mathbb{P}^n$ and $\mathrm{dim}X = 1$. (We do not assume that it is smooth or that $\Bbbk$ is algebraically closed.) Then $H^i\big(X, \mathcal{F}\big) = 0$ for $i > 1$. In this case,

$$\chi(\mathcal{F}) = \dim_{\Bbbk} H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F})$$

Assume that $\mathcal{F}$ is an invertible sheaf. Then $\mathcal{F} = \mathcal{L}(D)$ for some Cartier Divisor $D = \big[(U_i, f_i)\big]$ for $f_i \in \mathcal{K}(U)$ where K is the function field of X and $\mathcal{K}$ is the constant K sheaf. An

$$\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_{U_i} \cdot f_i^{-1}$$

Definition 3.48. D is an *effective Cartier divisor* of $f_i \in \mathcal{O}_X(U_i)$ for all i. In general, we have a decomposition $D = E - F$ where E and F is effective.

This means that if D is effective then $\mathcal{L}(-D) \subseteq \mathcal{O}_X$ is an ideal sheaf as it is locally generated by f_i. Hence, we have a short exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

where $D \subseteq X$ is the closed subscheme corresponding to the divisor D. (Note that this D may not be integral.)

Remark 3.49. If $D \neq X$ then as a topological space, it is a finite set of closed points as it is of dimension zero.

Eighth Lecture, 10th of March

Setup: Let $X \hookrightarrow \mathbb{P}^n_{\Bbbk}$ be a closed integral subscheme of dimension 1 (over Spec$\Bbbk$). The goal is to study invertible sheaves on these projective curves.

Let $D \in \mathrm{CaDiv}(X)$ i.e. $D = [(U_i, f_i)]$. We defined $\mathcal{L}(D)$ as the invertible sheaves associated to D. Over U_i this sheaf is generated by f_i^{-1}. Moreover, a divisor D is called effective if there exists a representation (U_i, f_i) such that $f_i \in \mathcal{O}(U_i)$. It is known that any Cartier divisor can be decomposed as a difference of effective divisors since any rational function is the quotient of two regular ones.

Assume D is effective, $\mathcal{L}(-D) \subseteq \mathcal{O}_X$ is a coherent ideal sheaf corresponding to a closed subscheme not equal to X. We denote this associated closed subscheme by the same D.

As $\dim D = 0$ we get that $D = \mathrm{Spec}R$ where R is a finite dimensional $\Bbbk$-algebra (since zero dimensional implies Artinian which – over $\Bbbk$ – implies finite dimensional). Hence, we may decompose $\mathrm{Spec}R = \cup_i \mathrm{Spec}R_i$ where R_i are local finite dimensional $\Bbbk$-algebras.

Definition 3.50. Assume that D is effective. Then

$$\deg D := \sum_{p \in D \text{ closed}} [\kappa(p) : \Bbbk] \cdot m_p(D)$$

where $m_p(D) := \dim_{\kappa(p)} \mathcal{O}_{X,p}/(f)$ using a local equation $f \in \mathcal{O}(U_p)$ for D (U_p is some affine open neighborhood of p). For general D, write $D = E - F$ where E and F are effective. Then we define $\deg D := \deg E - \deg F$. One can check that this degree is well-defined.

Remark 3.51. In dimension 1, we have

$$\chi\big(\mathcal{L}(D)\big) := \dim_{\Bbbk} H^0\big(X, \mathcal{L}(D)\big) - \dim_{\Bbbk} H^1\big(X, \mathcal{L}(D)\big) \tag{3.1}$$

Theorem 3.52. (Riemann-Roch) *For all Cartier-divisor $D \in \mathrm{CaDiv}(X)$,*

$$\chi\big(\mathcal{L}(D)\big) - \chi(\mathcal{O}_X) = \deg D$$

Remark 3.53. Another formulation: We define $g := \dim_{\Bbbk} H^1(X, \mathcal{O}_X)$ called the arithmetic genus. Then, by Eq. 3.1 applied for $\mathcal{L}(D) = \mathcal{O}_X$ ($D = 0$), we have

$$\chi\big(\mathcal{L}(D)\big) = \deg(F) - g + 1$$

as $\dim_{\Bbbk} H^0(X, \mathcal{O}_X) = \dim_{\Bbbk} \Gamma(X, \mathcal{O}_X) = 1$ since X is projective.

Corollary 3.54. (Riemann) $\dim H^0\big(X, \mathcal{L}(D)\big) \geq \deg D - g + 1$.

Remark 3.55. The term $\dim H^0\big(X, \mathcal{L}(D)\big)$ was called $L(D)$ last year.

Corollary 3.56. *For $f \in K(X)^\times$ we have $\deg\big(\mathrm{div}(f)\big) = 0$.*

Proof. $\mathcal{O}_X \cong \mathcal{L}\big(\mathrm{div}(f)\big) \leq \mathcal{K}(X)$ using the morphism $g \mapsto gf^{-1} \in K(X)$. We conclude that $\deg\big(\mathrm{div}(f)\big) = \deg\big(\mathrm{div}(1)\big) = 0$ as $\mathcal{L}(\mathrm{div}(1)) = \mathcal{O}_X$. □

Lemma 3.57. *Let S be a projective scheme over $\Bbbk$ and $S \hookrightarrow \mathbb{P}^n_{\Bbbk}$ a closed subscheme. Assume that we have an exact sequence of coherent sheaves*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

then $\chi(\mathcal{F}) + \chi(\mathcal{H}) = \chi(\mathcal{G})$.

Proof. We know that the alternating sum of the dimensions of the corresponding long exact sequence of homologies (that has only finitely many nonzero term by Theorem 3.23 and every term is finite dimensional by Theorem 3.34) is zero. Rearranging the terms give the statement. □

Proof of Theorem 3.52. Let $D = E - F$ where E and F are effective.

$$0 \to \mathcal{L}(-F) \to \mathcal{O}_X \to i_*\mathcal{O}_F \to 0$$

where $i : F \hookrightarrow X$ is the inclusion. Then, take its tensor product with $\mathcal{L}(E)$ giving

$$0 \to \mathcal{L}(D) \to \mathcal{L}(E) \to i_*i^*\mathcal{L}(E) \to 0 \tag{3.2}$$

which is again exact as $\mathcal{L}(E)$ is locally free of rank 1, hence on the level of stalks it is flat and exactness can be checked on stalks. Assume that F is affine. Then $H^i\big(F, i^*\mathcal{L}(E)\big) = 0$ for all $i > 0$ by Serre's vanishing theorem 3.15, but $H^i\big(F, i^*\mathcal{L}(E)\big) \cong H^i\big(X, i_*i^*\mathcal{L}(E)\big)$ by Corollary 3.17. (This fact can be shown without the vanishing theorem too.)

We claim that we have

$$\chi\big(i^*\mathcal{L}(E)\big) = \dim_{\Bbbk} H^0\big(F, i^*\mathcal{L}(D)\big) = \deg F$$

Indeed, the first equality is clear by dimensions, and $i^*\mathcal{L}(E) = \oplus_{p\in F}\mathcal{L}(E)_p$ by dimensions again. Moreover, $\oplus_{p\in F}\mathcal{L}(E)_p \cong \oplus_{p\in F}\mathcal{O}_{F,p}$ as it is locally free. Therefore,

$$\dim_{\Bbbk} H^0\big(F, i^*\mathcal{L}(D)\big) = \sum \dim_{\Bbbk}\mathcal{O}_{F,p} = \deg F$$

Now, we may apply the lemma to Equation 3.2 giving

$$\chi\big(\mathcal{L}(D)\big) = \chi(\mathcal{L}(E)) - \chi\big(i_*i^*\mathcal{L}(E)\big)$$

where we already computed that $\chi\big(i_*i^*\mathcal{L}(E)\big) = \deg F$.

Applying the argument for $D = 0$, $E = F$ we obtain

$$\chi\big(\mathcal{O}_X\big) = \chi(\mathcal{L}(E)) - \deg E$$

Together these two give $\chi\big(\mathcal{L}(D)\big) - \chi(\mathcal{O}_X) = \deg E - \deg F = \deg D$. □

Remark 3.58. The hard part of the proof is that the cohomologies are finite dimensional. Other than that, it is a small computation of exact sequences.

3.5 Ample invertible sheaves

Setup: Let $S = \mathrm{Spec} A$ where A Noetherian, let $X \to S$ be a proper morphism and let $\mathcal{L}$ be a line bundle on X.

Remark 3.59. Recall that $X \to S$ is called proper if it is separated, of finite type and universally closed i.e. $X \times_S Y \to Y$ is closed for all morphisms $Y \to S$. We know that the following maps are proper:

- $\mathbb{P}^n \to S$
- closed immersions
- if f and g are proper then $f \circ g$ is proper too

Definition 3.60. Under the given hypotheses, $\mathcal{L}$ is called *very ample* if there exists a closed embedding $X \hookrightarrow \mathbb{P}^N_S$ over S such that $i^*\mathcal{O}(1) \cong \mathcal{L}$.

Theorem 3.61. *The following are equivalent for $\mathcal{L}$:*

1. *There exists an $m > 0$ such that $\mathcal{L}^{\otimes m}$ is very ample.*
2. *For all coherent sheaves $\mathcal{F}$ on X there exists an $n_0 = n_0(\mathcal{F}) \geq 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.*
3. *For all coherent sheaves $\mathcal{F}$ on X there exists an $n_0 = n_0(\mathcal{F}) \geq 0$ such that for all $n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $i > 0$.*

Remark 3.62. Note that the first one depends on S while the second and third do not.

Definition 3.63. If $\mathcal{L}$ satisfies any of the conditions (hence all) then $\mathcal{L}$ is called *ample*.

Proof. We have seen 1) $\Rightarrow$ 3) last time since $\mathcal{L}^{\otimes m} = i^*\mathcal{O}(1)$ for $i : X \hookrightarrow \mathbb{P}^N_S$ then

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \cong H^i\big(\mathbb{P}^N_S, i_*(\mathcal{F} \otimes i^*(\mathcal{O}(1))\big) \cong H^i\big(\mathbb{P}^N_S, (i_*\mathcal{F})(1)\big)$$

Even more generally,

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m \cdot r}) = H^i(\mathbb{P}^N_S, (i_*\mathcal{F})(r)\big) = 0$$

for big enough r by Serre's theorem since $i_*\mathcal{F}$ is coherent on $\mathbb{P}^N_S$.

Next, we prove 3) $\Rightarrow$ 2): Let $p \in X$ be a closed point, and let $\mathcal{J}_p$ be the ideal sheaf of the closed subscheme $\{p\} \hookrightarrow X$. Then, we have an exact sequence

$$0 \to \mathcal{J}_p \to \mathcal{O}_X, \to \kappa(p) \to 0$$

where $\kappa(p)$ is the skyscraper sheaf with the ring $\kappa(p)$ above p. Take a coherent sheaf $\mathcal{F}$ then tensor the above equation with $\mathcal{F}$:

$$0 \to \mathcal{I}_p\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \kappa(p) \to 0$$

where $\mathcal{I}_p\mathcal{F} = \mathrm{Im}(\mathcal{I}_p \otimes \mathcal{F} \to \mathcal{F})$. Note that if $\mathcal{F}$ is a coherent sheaf then $\mathcal{I}_p\mathcal{F}$ is coherent too. Then, tensoring it via $\mathcal{L}^{\otimes}$ (which is flat as it is locally free) we get

$$0 \to \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \kappa(p) \to 0$$

It has a long exact sequence of cohomologies:

$$\dots \to H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \kappa(p)) \to H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \dots$$

By the assumption, $H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for high enough n hence the above map is surjective. However, we have

$$\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \kappa(p) \cong (\mathcal{F} \otimes \mathcal{L}^n)_p \otimes_{\mathcal{O}_{X,p}} \kappa(p) \cong (\mathcal{F} \otimes \mathcal{L}^{\otimes n})_p / m_p(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_p$$

Hence, we know that the image of the finitely generated module $H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ generates $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_p$ modulo its maximal ideal so – by Nakayama's lemma – it must generate $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_p$ itself, i.e. the stalk is generated by global sections for this n. Unfortunately, the n may depend on p (and on $\mathcal{F}$ but that is not a problem). Moreover, we do not know what happens after n.

The proved property means that there exists a neighborhood of p such that for all $p \in U$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. Apply this to $\mathcal{F} = \mathcal{O}_X$: we get that there exists an n_1 such that $\mathcal{L}^{\otimes n_1}$ is generated by global sections in a neighborhood V of p. Then, for all $0 \leq r \leq n_1$ there exists a neighborhood U_r of p such that $\mathcal{F} \otimes \mathcal{L}^{\otimes(n_0+r)}$ is generated by global section, if n_0 is chosen sufficiently large. Then on $\cap_r U_r \cap V$ we get that $\mathcal{F} \otimes \mathcal{L}^{\otimes(n_0+r)} \otimes (\mathcal{L}^{\otimes n_1})^m$ is generated by global section for all m. Hence, by division by remainder, we get that $\mathcal{F} \otimes \mathcal{L}^{\otimes M}$ on $\cap_r U_r \cap V$ is generated by global sections for all $M > 0$.

Now, to eliminate the p-dependency, we may use that X is compact. [?]

Direction 2) $\Rightarrow$ 1): for any $p \in X$ there exists an affine open neighborhood U or p such that $\mathcal{L}|_U$ is free on U. Take $Y = X \backslash U$ which we consider as a closed subscheme of X with the reduced subscheme structure. This corresponds to a coherent sheaf of ideals $\mathcal{I}_Y \subseteq \mathcal{O}_X$. By the assumptino $\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for sufficiently large n i.e. there exists an $s \in H^0(X, \mathcal{J}_Y \otimes \mathcal{L}^{\otimes n})$ such that $s_p \notin m_p\big(\mathcal{J}_Y \otimes \mathcal{L}^{\otimes n}\big)_p$ as $\mathcal{I}_{Y,p} \cong \mathcal{O}_p$ hence $\big(\mathcal{J}_Y \otimes \mathcal{L}^{\otimes n}\big)_p$ is generated by one section.

Let $X_S := \{p \in U \mid s_p \notin m_p \mathcal{L}^{\otimes n}\}$. In fact, it is an affine open subset of U. Indeed, $\mathcal{L}^{\otimes n}|_U \cong \mathcal{O}_U$ hence $s|_U$ defines an element of $\mathcal{O}_U(U)$. This way we get an affine open covering of X.

Lemma 3.64. *Let X be a scheme, $\mathcal{L}$ an invertible sheaf and $\mathcal{F}$ is a quasi-coherent sheaf. Assume that X has a finite open covering, by affine opens such that U_i such that $\mathcal{L}|_{U_i}$ is free for all i and $U_i \cap U_j$ is compact for all i, j. If $t \in \mathcal{F}(X_f)$ where $X_f := \{p \in X \mid f_p \notin m_p \mathcal{L}_p\}$ for some $f \in \Gamma(X)$ then there exists an $n > 0$ such that $f^n t$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.*

Apply this for $f = t = s$. It allows us to extend s to a global section.[?] In details, we get an affine open covering $\{X_{s_i}\}_i$ such that $s_i \in H^0(X, \mathcal{L}^{\otimes m})$ where X_{s_i} is defined as above. Now, we repalce $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ (as $\mathcal{L}^{\otimes m}$ also satisfies the assumptions in 2)). So we may assume that $m = 1$. By properness we may assume that $B_i := H^0(X_{s_i}, \mathcal{O}_{X_i})$ is a finitely generated A-algebra. Let $\{b_{ij} \mid j = 1, \ldots, k\}$ is a set of generators of B_i as an A-algebra, for all i. By the lemma, for all i, j there exists an n_{ij} such that $s_i^{n_{ij}} b_{ij}$ extends to $c_{ij} \in H^0(X, \mathcal{L}^{\otimes n_{ij}})$. Choosing n_{ij} big enough, we may assume that they all equal to n. Then we get a morphism

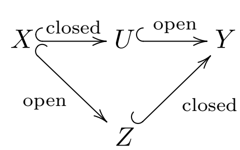
$$A[y_j, y_{ij} \mid 1 \leq j \leq r_i] \to B_i$$

$y_{ij} \mapsto b_i$ where $y_j := s_j|_{X_{s_i}}/s_i$. These define a morphism $X \to \mathbb{P}^N$ where $N = n + nr_i - 1$ by gluing the corresponding closed embeddings $X_{s_i} \hookrightarrow D_+(x_i) \subseteq \mathbb{P}^N$. Unfortunately, we only get

$$X \overset{\text{closed}}{\hookrightarrow} \cup D_+(x_i) \overset{\text{open}}{\hookrightarrow} \mathbb{P}^N$$

where $\cup D_+(x_i)$ not necessarily equal to $\mathbb{P}^N$ as there are other variables too. This is solved by the following lemma:

Lemma 3.65. *Assume that given a setup $X \overset{\text{closed}}{\hookrightarrow} U \overset{\text{open}}{\hookrightarrow} Y$ where Y is locally Noetherian. Then there exists a factorization*



Applying this lemma on our situation and useing that X is proper over A we get that $X \to Z$ is open and closed hence an isomorphism. This way we get a closed embedding $X \hookrightarrow \mathbb{P}^n_A$.

Sketch of the proof of the Lemma 3.65. Consider the presheaf $V \mapsto \operatorname{Ker}\big(\mathcal{O}_X(V) \to \mathcal{O}_Y(Y \cap V)\big) =: I(V)$. We claim that I is a quasi-coherent idealsheaf on X. Indeed, it is clearly an ideal sheaf and for any open immersion $W \cong \operatorname{Spec} A \hookrightarrow X$ one has

$$I|_W = \operatorname{Ker}\big(A \to \widetilde{\mathcal{O}_Y\big(Y \cap W\big)}\big)$$

Hence, it is indeed quasi-coherent. By the claim, we get a corresponding closed subscheme $Z \hookrightarrow X$ where $Y = Z \cap U$ by construction. So $Y \subseteq Z$ is open. □

□

Nineth Lecture, 17th of March

Setup: Let $X \to \operatorname{Spec} A$ be a proper morphism where A is Noetherian and $\mathcal{L}$ an invertible sheaf on X. We defined $\mathcal{L}$ to be ample if there exists $m > 0$ and a closed immersion $X \to \mathbb{P}^n_A$ over A such that $\mathcal{L}^{\otimes m} \cong i^*\mathcal{O}(1)$. For this notion, last time we proved a cohomological characterization: for all coherent sheaves $\mathcal{F}$ on X there exists an n_0 such that for all $n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$.

3.6 Normalization of a projective scheme

Definition 3.66. An integral scheme X is *normal* if and only if $\mathcal{O}_{X,p}$ is integrally closed for all $P \in X$.

Definition 3.67. For an integral scheme X, a *normalization* $X^\nu \to X$ is a morphism such that for all $Y \to X$ dominant maps where Y is integral, normal, there exists a factorization

$$\begin{array}{ccc} Y & \longrightarrow & X^\nu \\ & \searrow & \Big\downarrow {\scriptstyle \exists!} \\ & & X \end{array}$$

Remark 3.68. Note that a morphism $Y \to X$ being dominant between integral schemes is equivalent to the fact that it sends the generic point of Y into the generic point of X.

Lemma 3.69.

1. *If $X^\nu \to X$ exists, it is unique up to unique isomorphism.*
2. *If X is affine then X^ν exists, namely for $X = \operatorname{Spec} A$, $X^\nu := \operatorname{Spec}\overline{A}$ where $\overline{A}$ is the integral closure of A.*

Corollary 3.70. *X^ν exists in general by patching together the affine normalizations.*

Remark 3.71. If X is of finite type over a field then $X^\nu \to X$ is finite. (By the nontrivial theorem of finiteness of the integral closure.) Unfortunately, it is not always finite.

Theorem 3.72. *Let $X \hookrightarrow \mathbb{P}^n_{\Bbbk}$ be an integral closed subscheme over the field $\Bbbk$. Then X^ν has a closed embedding in some $\mathbb{P}^N_{\Bbbk}$.*

Proof. By the remark, $X^\nu \xrightarrow{p} X$ is finite (hence proper), and $X \to \operatorname{Spec}\Bbbk$ is also proper by Remark 3.59. So their composition is also proper. Then we may take an ample line bundle $\mathcal{L}$ on X (e.g. take $\mathcal{L} = i^*\mathcal{O}(1)$). We claim that $p^*\mathcal{L}$ is also ample. That is enough to prove the theorem by Theorem 3.17. The ampleness is checked by Serre's criterion, Theorem 3.61, that was proved last time.

To check Serre's criterion, let $\mathcal{F}$ a coherent sheaf on X^ν. We have to prove that $H^i(X^\nu, \mathcal{F} \otimes p^*\mathcal{L}^{\otimes m})$ is zero. However, by p finite, in particular it is affine, we get

$$H^i(X^\nu, \mathcal{F} \otimes p^*\mathcal{L}^{\otimes m}) \cong H^i\big(X, p_*(\mathcal{F} \otimes p^*\mathcal{L}^{\otimes m})\big) \overset{3.37}{\cong} H^i\big(X, (p_*\mathcal{F}) \otimes \mathcal{L}^{\otimes m}\big)$$

Note here that as $\mathcal{F}$ is coherent and p is finite, $p_*\mathcal{F}$ is also coherent. However, as $\mathcal{L}$ is ample, the right hand side is zero for every big enough m. □

4 Differential forms

Definition 4.1. Let B be a ring, M a B-module. A *derivation* $\mathrm{d} : B \to M$ is a map such that

- $\mathrm{d}(x + y) = d(x) + d(y)$ for all $x, y \in M$, and
- (Leibniz's rule) $d(xy) = x\mathrm{d}y + y\mathrm{d}x$

If, moreover, B is an A-algebra, then d is an A-derivation if it is A-linear (or equivalently, $\mathrm{d}(A) = 0$).

Example 4.2. Assume that B is an A-algebra such that B is isomorphic to $A \oplus I$ as an A-module where $I \lhd B$ is an ideal with $I^2 = 0$. Then, the projection $\mathrm{d} : B \to I$ is an A-derivation. Indeed, $\mathrm{d}(A)$ is zero by definition, it is additive as it is a module map and if $x_i = a_i + \mathrm{d}x_i$ for $x_i \in B$, $a_i \in A$ $(i = 1, 2)$ then we have

$$\mathrm{d}(x_1x_2) = \mathrm{d}\big((a_1 + \mathrm{d}x_1)(a_2 + \mathrm{d}x_2)\big) = d(a_1a_2) + a_2\mathrm{d}x_1 + a_1\mathrm{d}x_2 + 0$$

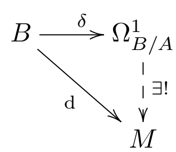
as we stated.

Definition 4.3. For all B-modules M, let

$$\mathrm{Der}_A(B, M) = \{A\text{-derivations } B \to M\}$$

the set of derivations. It is naturally a B-module.

Proposition 4.4. *Let B be an A-algebra. There exists a B-module $\Omega^1_{B/A}$ and a derivation $\delta : B \to \Omega^1_{B/A}$ satisfying the universal property such that for all A-derivations $d : B \to M$ there exists a unique B-module homomorphism completing the following commutative diagram:*



Definition 4.5. $\Omega^1_{B/A}$ is the module of *relative differentials* (or differential forms) of B over A.

Proof. Let F be a free B-module on the symbols $\{\mathrm{d}b \mid b \in B\}$. Define

$$\Omega^1_{B/A} := F/\langle \mathrm{d}(a_1b_1 + a_2b_2) - a_1\mathrm{d}b_1 - a_2\mathrm{d}b_2,\ \mathrm{d}(b_1b_2) - b_1\mathrm{d}b_2 - b_2\mathrm{d}b_1 \mid a_1, a_2 \in A,\ b_1, b_2 \in B\rangle$$

and $\delta(b) = \mathrm{d}b$. Then the map $\Omega^1_{B/A} \to M$ is induced by $F \ni \mathrm{d}b \mapsto \delta(b)$. As δ satisfies the given relations, it really factors through $\Omega^1_{B/A}$. □

Example 4.6. Assume that $B = A[x_1, \dots, x_n]/(f_1, \dots, f_m)$. Then

$$\Omega^1_{B/A} = \bigoplus_{i=1}^{n} B\mathrm{d}x_i/\Big\langle \sum_j \partial_j f_i \mathrm{d}x_j \mid i = 1, \dots, m\Big\rangle$$

Indeed, if $B = A[x_1, \dots, x_n]$ then $\Omega^1_{B/A} = \oplus B\mathrm{d}x_i$ as it already satisfies the universal property. In general, $\mathrm{Der}_A(B, M) = \{\delta \in \mathrm{Der}_A(A[x_1, \dots, x_n], m_i) \mid \delta(f_i) = 0\}$. Equivalently, $\delta(f_i) = 0$ means that $\sum_j \partial_j f_i \mathrm{d}x_j = 0$. In short, for finitely presented algebras, one has an easier description of $\Omega^1_{B/A}$.

Lemma 4.7.

1. (Localization) *If* $S \subseteq B$ *a multiplicatively closed set then* $\Omega^1_{B_S/A} \cong \Omega^1_{B/A} \otimes_B B_S$.
2. (Base change) *If* $A \to A'$ *is a homomorphism then* $\Omega^1_{B \otimes_A A'/A'} \cong \Omega^1_{B/A} \otimes_B (B \otimes_A A') \cong \Omega^1_{B/A} \otimes_A A'$

Proof. To see 1), note that given a $\delta \in \mathrm{Der}_A(B, M)$ then it extends uniquely to $\delta_S \in \mathrm{Der}_A(B_S, M \otimes_B B_S)$ by defining

$$\delta_S\left(\frac{b}{s}\right) := (\delta(b)s - b\delta(s)) \otimes \frac{1}{s^2}$$

This applies to the fixed $\mathrm{d} : B \to \Omega^1_{B/A}$ giving $\mathrm{d}_S : B_S \to \Omega^1_{B/A} \otimes_B B_S$. Now, one can check that this pair $(\mathrm{d}_S, \Omega^1_{B/A} \otimes_B B_S)$ satisfies the universal property.

For part 2), the argument is similar: $\mathrm{d} : B \to \Omega^1_{B/A}$ gives rise to $\mathrm{d}' : B \otimes_A A^{'} \to \Omega^1_{B/A} \otimes_A A'$ that also satisfies the universal property. □

Proposition 4.8. *Let* $\Bbbk$ *be a perfect field,* B *a localization of a finitely generated* $\Bbbk$*-algebra at some maximal ideal* P*. Then*

1. *There exists a canonical isomorphism* $\Omega^1_{B/\Bbbk}/P\Omega^1_{B/\Bbbk} \cong P/P^2$.
2. *If* B *is an integral domain, then* $\Omega^1_{B/\Bbbk}$ *is free of rank* $n = \dim B$ *if and only if* $\dim_\Bbbk P/P^2 = n$.

Remark 4.9. It is not true when $\Bbbk$ is not perfect. Even if $\Bbbk$ is perfect, $\Omega^1_{B/\Bbbk}$ is not necessarily free in general.

Proof. We only prove for the case when $\Bbbk$ is algebraically closed. (The proof of the general case in the lecture notes uses the Cohen structure theorem...) In this case, $B/P \cong \Bbbk$. For the first part, note that $\Omega^1_{B/\Bbbk}/P\Omega^1_{B/\Bbbk}$ satisfies the universal property for every $\Bbbk$-module M. Indeed, if we view M as a B-module (via $B \to B/P \cong \Bbbk$) we get that for all $d \in \mathrm{Der}_\Bbbk(B, M)$, there exists a unique factorization of the stated form. We claim that P/P^2 satisfies the same universal property.

Notice that $\delta(P^2) = 0$ by the Leibniz's rule and by $P \cdot M = 0$. Therefore, replacing B by B/P^2, we may assume that $P^2 = 0$. Then $B \cong \Bbbk \oplus P$ as a $\Bbbk$-module and $P^2 = 0$. We can also define the corresponding derivation $B \to P$ by projection. Since $\delta(\Bbbk) = 0$, we get that δ really factors through this projection $B \to P$.

For part 2), assume that B is now an integral domain and take $K = \mathrm{Frac}(B)$ so $n = \dim(B) = \mathrm{trdeg}(K \mid \Bbbk)$. We need the following lemma.

Lemma 4.10. $\Omega^1_{K/\Bbbk} \cong \oplus_{i=1}^n K\mathrm{d}x_i$ *for some* $x_i \in K$.

Proof. By the separable generation of finitely generated field extensions, there exists $x_1, \ldots, x_n$ such that $K \mid \Bbbk(x_1, \ldots, x_n)$ is finite and separable (We use this without proof.). It follows that $K = \mathrm{Frac}\big(\Bbbk[x_1, \ldots, x_n, x]/(f)\big)$ where $\partial_x f = 0$. By the base change property (Lemma4.7)

$$\Omega^1_{K/\Bbbk} \cong \left(\Omega^1_{\Bbbk[x_1,\ldots,x_n,x]/(f) \big/ \Bbbk}\right) \otimes_\Bbbk K$$

i.e. we have $n + 1$ generators and one nonzero relations. □

Direction $\Rightarrow$ of 2) is clear by part 1), if $\Omega^1_{B/\Bbbk}$ is free of rank n then modulo $P\Omega^1_{B/\Bbbk}$, it gives a vector space of dimension n. Conversely, we know that

$$\dim_\Bbbk \left(\Omega^1_{B/\Bbbk}/P\Omega^1_{B/\Bbbk}\right) = n$$

hence $\Omega^1_{B/\Bbbk}$ is a finitely generated B-module by n elements, by Nakayama's lemma. Let us denote these elements by $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$. We need to prove that we have no relation among these.

By the localization property, Lemma 4.7, $\Omega^1_{K/\Bbbk} \cong \Omega^1_{B/\Bbbk} \otimes_B K$ as K-vector spaces with basis $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$ (by the Lemma). If there were a relation with coefficients in B then we would obtain a relation on the right hand side, hence on the left hand side, but there is none. □

Remark 4.11. The fact that $\Omega^1_{\mathcal{O}_{X,p},\Bbbk}$ is free of rank $\dim \mathcal{O}_{X,p}$ is exactly the non-vanishing of the Jacobi criterion. (So this proposition is a generalization of the fact that the smoothness of a point can be characterized by the Jacobi criterion.) Indeed, let B be a localization of $\Bbbk[x_1,\dots,x_N]/(f_1,\dots,f_m)$. Then $\Omega^1_{B/\Bbbk}$ is free of rank n if and only if among the equalities $\sum_j \partial_j f_i \mathrm{d}x_j = 0$ there are $N-n$ independent equations, i.e. the Jacobian determinant is nonzero.

Definition 4.12. Now, we define for a scheme $X \to \mathrm{Spec}\Bbbk$ over a field $\Bbbk$ a quasi-coherent sheaf $\Omega^1_{X/\Bbbk}$ as follows: If $X = \mathrm{Spec}A$ where A is a $\Bbbk$-algebra then $\Omega^1_{X/\Bbbk} = \widetilde{\Omega^1_{A/\Bbbk}}$. In general case, patch using the localization property of $\Omega^1_{A/\Bbbk}$ i.e. that for $f \in A$ we have $\Omega^1_{A_f/\Bbbk} \cong \Omega^1_{A/\Bbbk} \otimes_A A_f$.

Remark 4.13.

1. In general, it is possible to define $\Omega^1_{X/S}$ for an arbitrary morphism of schemes $X \to S$. When $X \to S$ is an affine morphism, the same gluing argument works (via using localization and base change properties). If it is not affine, then one has to use another idea called infinitesimal thickenings. For details, see references.

2. If X is of finite type over $\Bbbk$ then $\Omega^1_{X/\Bbbk}$ is coherent.

Definition 4.14. $X \to \mathrm{Spec}\Bbbk$ is *smooth* if and only if $\Omega^1_{X/\Bbbk}$ is locally free. In this case, $\dim X = \mathrm{rk}\Omega^1_{X/\Bbbk}$.

Corollary 4.15. *If X is a scheme over a perfect field $\Bbbk$ then it is smooth over (of dimension n) if and only if $\mathcal{O}_{X,p}$ is a regular local ring (of dimension n).*

Definition 4.16. Let $X \to \mathrm{Spec}\Bbbk$ be smooth of dimension n. Then $\omega_{X/\Bbbk} := \Lambda^n \Omega^1_{X/\Bbbk}$ is a locally free sheaf of rank 1, called the *canonical sheaf* of $X/\Bbbk$.

Question 4.17. *Assume that $X \to \mathrm{Spec}\Bbbk$ is a smooth scheme of dimension n and $C \overset{i}{\hookrightarrow} X \to \mathrm{Spec}\Bbbk$ is a smooth, closed subscheme of codimension 1. How can we relate $\Omega^1_{C/\Bbbk}$ and $\Omega^1_{X/\Bbbk}$? What is with the relation of $\omega_{C/\Bbbk}$ and $\omega_{X/\Bbbk}$?*

Proposition 4.18. *In the above situation, there exists an exact sequence of locally free sheaves on C:*

$$0 \to \mathcal{L}(-C) \otimes_{\mathcal{O}_X} i_*\mathcal{O}_C \to \Omega_{X/\Bbbk} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_C \to \Omega^1_{C/\Bbbk} \to 0$$

Corollary 4.19. $\omega_{C/\Bbbk} \cong i^*\big(\omega_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{L}(C)\big)$

Proof. Recall the general fact that if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves of rank n', n and n'' respectively, then there is a canonical isomorphism

$$\Lambda^{n'}\mathcal{E}' \otimes_{\mathcal{O}_X} \Lambda^{n''}\mathcal{E}'' \cong \Lambda^n \mathcal{E}$$

(Its proof can be reduced to the case of free modules $0 \to E' \to E \to E'' \to 0$ over a ring. There, one can take compatible bases: $(e_i)_{i \le n'}$ for E', $(\bar{f}_j)_{j \le n''}$ for E'' and lifts $(f_j)_{j \le n''}$for E. Then, define

$$e_1 \wedge \cdots \wedge e_{n'} \otimes \bar{f}_1 \wedge \cdots \wedge \bar{f}_{n''} \mapsto (e_1 \wedge \cdots \wedge e_{n'} \wedge f_1 \wedge \cdots \wedge f_{n''})$$

One can check that it is well defined and an isomorphism of free modules of rank 1.)

In our proposition, it gives us

$$\omega_{C/\Bbbk} \cong \omega_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{O}_C \cong \omega_{C/\Bbbk} \otimes \big(\mathcal{L}(-C) \otimes_{\mathcal{O}_X} \mathcal{O}_C\big)$$

so by tensoring via $\mathcal{L}(C) \otimes \mathcal{O}_C$ it gives the statement (using the fact that i^* is the same as $\otimes \mathcal{O}_C$). ☐

Lemma 4.20. (Second fundamental sequence) *Let* $0 \to I \to A \to B \to 0$ *be an exact sequence of (non-unital)* $\Bbbk$*-algebras. Then we have an associated exact sequence*

$$I/I^2 \to \Omega^1_{A/\Bbbk} \otimes_{\Bbbk} B \to \Omega^1_{B/\Bbbk} \to 0$$

of B*-modules using the maps* $\delta(x \bmod I^2) \mapsto \mathrm{d}_{A/\Bbbk}(x) \otimes 1$ *where* $\mathrm{d}_{A/\Bbbk} : A \to \Omega^1_{A/\Bbbk}$.

Proof. By patching argument, it is enough to show that for all B-modules M

$$0 \to \mathrm{Der}_{\Bbbk}(B, M) \to \mathrm{Der}_{\Bbbk}(A, M) \xrightarrow{\delta^*} \mathrm{Hom}_B(I/I^2, M)$$

is an exact sequence of B-modules, where δ^* is the restriction from A to I factored through I/I^2. Indeed, $\mathrm{Der}_{\Bbbk}(B, M), \cong \mathrm{Hom}_B(\Omega^1_{B/\Bbbk}, M)$ and

$$\mathrm{Der}_{\Bbbk}(A, M) \cong \mathrm{Hom}_B(\Omega^1_{A/\Bbbk} \otimes_A B, M) \cong \mathrm{Hom}_A(\Omega^1_{A/\Bbbk}, M)$$

by definition. By general homological algebra, the mentioned sequence is exact if and only if it is exact on every $\mathrm{Hom}(., M)$.

However, by definition $\delta^*(D) = 0$ if and only if $D(I) = 0$. □

Proof of Proposition 4.18. By previous discussion closed subscheme C corresponds to a quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. This ideal sheaf is exactly $\mathcal{L}(-C)$. It can be summarized in the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

which locally corresponds to the morphism of (non-unital) rings $0 \to I \to A \to B \to 0$.

By the lemma, there exists an exact sequence of coherent sheaves on C:

$$\mathcal{J}/\mathcal{J}^2 \to \Omega^1_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{O}_C \to \Omega^1_{C/\Bbbk} \to 0$$

However, the first term is $\mathcal{J}/\mathcal{J}^2 = \mathcal{I} \cdot (\mathcal{O}_X/\mathcal{J}) \cong \mathcal{L}(-C) \cdot \mathcal{O}_C \cong \mathcal{L}(-C) \otimes_{\mathcal{O}_X} \mathcal{O}_C$. Therefore, we almost got the proposition, except the injectivity of $\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{O}_C$ on the left hand side.

To prove this last part, we have to use the fact that X and C are smooth, i.e. $\Omega^1_{X/\Bbbk}$ is locally free of rank n and $\Omega^1_{C/\Bbbk}$ is locally free of rank $n-1$. Therefore, it splits (locally) so $\mathrm{Ker}(\alpha)$ is locally free of rank 1. But $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{L}(-C) \otimes_{\mathcal{O}_X} \mathcal{O}_C$ is also locally free of rank 1, so $\mathcal{I}/\mathcal{I}^2 \to \ker(\alpha)$ is a surjective morphism of locally free sheaves of rank 1, i.e. it is an isomorphism. (Indeed, on the level of stalks $\mathcal{I}/\mathcal{I}^2$ and $\mathrm{Ker}(\alpha)$ are both free of rank 1 hence the surjection splits but $\mathcal{O}_{X,p}$ is Noetherian local so by tensoring via $\mathcal{O}_{X,p}/m\mathcal{O}_{X,p}$, we get that the kernel must be zero. □

Tenth Lecture, 24th of March

4.1 Serre duality

Theorem 4.21. (Serre duality) *Let* $\Bbbk$ *be a perfect field and* $X \to \mathrm{Spec}\Bbbk$ *a smooth projective morphism, where* X *is integral of dimension* d*. Then*

- $H^d(X, \omega_{X/\Bbbk}) \xrightarrow{\cong} \Bbbk$
- *There exists a perfect (i.e. non-degenerate) pairing of finite dimensional* $\Bbbk$*-vector spaces*

  $$H^i(X, \mathcal{F}) \times H^{d-i}(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_{X/\Bbbk}) \to H^d(X, \omega_{X/\Bbbk}) \cong \Bbbk$$

  for any locally free sheaf $\mathcal{F}$ *on* X *for all* $0 \leq i \leq d$.

Remark 4.22. The first part is harder. Moreover, the perfectness of $\Bbbk$ is assumed so that we do not have to distinguish between regularity and smoothness.

Corollary 4.23. *If $d = 1$ and $\mathrm{rk}\mathcal{F} = 1$ then the pairing is*

$$H^0(X, \mathcal{F}) \times H^1(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_{X/\Bbbk}) \to H^1(X, \omega_{X/\Bbbk})$$

where $\omega_{X/\Bbbk} = \Omega^1_{X/\Bbbk}$ by definition.

Recall that by the cohomological Riemann-Roch formula for $\mathcal{L} = \mathcal{L}(D)$ we have

$$\dim_\Bbbk H^0(X, \mathcal{L}) - \dim_\Bbbk H^1(X, \mathcal{L}) = \deg_\Bbbk D - g + 1$$

By Serre's duality, we have

$$g = \dim_\Bbbk H^1(X, \mathcal{O}_X) = \dim_\Bbbk H^0(X, \Omega^1_{X/\Bbbk})$$

and $\dim_\Bbbk H^1(X, \mathcal{L}(D)) = \dim H^0(X, \mathcal{L}(K - D))$ where $\mathcal{L}(K) = \Omega^1_{X/\Bbbk}$ and K is called the canonical class. Putting all this together, we get

$$\dim H^0(X, \mathcal{L}(D)) - \dim_\Bbbk H^0(X, \mathcal{L}(K - D)) = \deg_\Bbbk D - g + 1$$

Note, that the original cohomological statement holds without the smoothness condition as well, but the latter form is the one we deduced last year for smooth projective curves.

Remark 4.24. The original proof of this statement by Serre goes by induction on dimension. Grothendieck generalized the statement where the proof uses a lot more technicality in the form that it proves that ω is a sheafification of some kind of Ext groups...

Sketch of Serre's approach. For $d = 1$, the proof can be found at Serre: Algebraic groups and class fields, Chapter I (1959). The whole proof for all d is hard to track down in the literature.

Curves:

construction of the pairing (called cup product) is the following: We start with the map

$$H^1(X, \mathcal{L}) \times \mathrm{Hom}(\mathcal{L}, \Omega^1_{X/\Bbbk}) \to H^1(X, \Omega^1_{X/\Bbbk})$$

using that H^1 is functorial in its arguments so we may apply an element of $\mathrm{Hom}(\mathcal{L}, \Omega^1_{X/\Bbbk})$ to a cocycle.

We prove that

$$\mathrm{Hom}(\mathcal{L}, \Omega^1_{X/\Bbbk}) \cong H^0(X, \mathcal{L}^\vee \otimes \Omega^1_{X/\Bbbk})$$

where $\mathcal{L}^\vee = \underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{O}_X)$, $U \mapsto \mathrm{Hom}(\mathcal{L}|_U, \Omega^1_{X/\Bbbk}|_U)$ is the (internal Hom or) Hom sheaf. Indeed, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{O}_X) \otimes \Omega^1_{X/\Bbbk} \overset{\cong}{\to} \underline{\mathrm{Hom}}(\mathcal{L}, \Omega^1_{X/\Bbbk})$$

defined by $\lambda \otimes \omega \mapsto (s \mapsto \lambda(s)\omega)$. This is an isomorphism on stalks since the sheaves are locally free of rank 1. Hence, we have

$$\mathrm{Hom}(\mathcal{L}, \Omega^1_{X/\Bbbk}) \cong H^0(X, \underline{\mathrm{Hom}}(\mathcal{L}, \Omega^1_{X/\Bbbk})) \cong H^0(X, \underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{O}_X) \otimes \Omega^1_{X/\Bbbk})$$

Definition 4.25. Assume that $\Bbbk = \overline{\Bbbk}$. We define the trace map $H^1(X, \Omega^1_{X/\Bbbk}) \to \Bbbk$ as follows:

Consider $K = K(X)$ as X is integral. By $\mathrm{tr.deg}(K \mid \Bbbk) = 1$ we have $\dim_K \Omega^1_{K/\Bbbk} = 1$. In fact, one can show that if $p \in X$ then we can choose a t such that it generates $M_p \subseteq \mathcal{O}_{X,p}$ and $K \mid \Bbbk(t)$ is separable hence t generates $\Omega^1_{K/\Bbbk}$ too. So if $\omega \in \Omega^1_{K/\Bbbk}$, we may write $\omega = g\mathrm{d}t$ for some $g \in K$.

As $\Bbbk = \overline{\Bbbk}$, we get an embedding of $\mathcal{O}_{X,p} \hookrightarrow \Bbbk[[t]]$ by completion argument since (t) is maximal and its residue field is $\Bbbk$. Similarly, we may embed $K = \mathrm{Frac}(\mathcal{O}_{X,p})$ into $\Bbbk((t))$, localizing the previous embedding. Let $g = \sum_{i=-n}^{\infty} a_i t^i$. We define as residue as $\mathrm{res}_p(\omega) = a_{-1}$ in complete analogy with complex geometry.

Proposition 4.26. *It is a nontrivial fact (that we do not prove) that*

- $\mathrm{res}_p(\omega)$ *does not depend on the choice of* t.
- $\sum_{p\in X} \mathrm{res}_p(\omega) = 0$ *where the sum is finite as the singular values of* g *is a proper closed set on the curve.*

Definition 4.27. Let Ω_K be the constant sheaf coming from $\Omega^1_{X/\Bbbk}$.

Proposition 4.28. *(Not proven, easy) There exists an exact sequence*

$$0 \to \Omega^1_{X/\Bbbk} \to \Omega_K \to \bigoplus_{p\in X \text{ closed}} i_{p*}\big(\Omega_K/\Omega^1_{(X/\Bbbk),p}\big) \to 0 \tag{4.1}$$

The corresponding long exact sequence of cohomologies give

$$\Omega_K \to \bigoplus_{p\in X} \Omega_K/\Omega^1_{K/\Bbbk} \to H^1(X,\Omega^1_X) \to H^1(X,\Omega_K)$$

where $H^1(X,\Omega_K)=0$ since Ω_K is flabby. However, we get a half-exact sequence by taking

$$\begin{array}{ccc} \Omega_K & \longrightarrow & \bigoplus_{p\in X} \Omega_K/\Omega^1_{K/\Bbbk} \\ & \searrow & \downarrow \sum \mathrm{res}_p \\ & & \Bbbk \end{array}$$

Using property 2 in Proposition 4.26 above. Moreover, by the surjectivity in Equation 4.1, the previous sequence, we get a factorization

$$\begin{array}{ccccc} \Omega_K & \longrightarrow & \bigoplus_{p\in X} \Omega_K/\Omega^1_{K/\Bbbk} & \longrightarrow & H^1(X,\Omega^1_X) \\ & \searrow & \downarrow \sum \mathrm{res}_p & \swarrow \mathrm{tr} & \\ & & \Bbbk & & \end{array}$$

This is the trace map. It is a hard theorem of Mittag-Leffler type that this trace map is injective. After that, duality follows without complication.

Generally:

Serre's construction of the trace map $H^d(X,\omega_{X/\Bbbk}) \xrightarrow{\cong} \Bbbk$. Assume that $d>1$ and the statement is true for $d-1$. We prove that

$$H^d(X,\omega_{X/\Bbbk}) \xrightarrow{\cong} H^{d-1}(C,\omega_{C/\Bbbk})$$

for a suitable smooth closed integral subscheme $C \hookrightarrow X$ of codimension 1. The way one can find such a smooth closed subscheme is by Bertini's theorem. It says that taking a general hypersurface section gives a smooth intersection. Let C be a subscheme chosen this way. Then we have the usual exact sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0$$

After tensoring it by $\omega_{X/\Bbbk}\otimes\mathcal{L}(C)$ we get

$$0 \to \omega_X \to \omega_X \otimes \mathcal{L}(C) \to i_*\mathcal{O}_C \otimes_{\mathcal{O}_X} \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}(C) \to 0 \tag{4.2}$$

The last term is

$$i_*\mathcal{O}_C \otimes_{\mathcal{O}_X} \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}(C) \cong i_*\big(i^*\omega_X \otimes \mathcal{L}(C)\big)$$

We have seen last time in Corollary 4.19 that $i^*\omega_X \otimes \mathcal{L}(C) \cong \omega_{C/\Bbbk}$. As C was an intersection of X with a hypersurface, we get that $\mathcal{L}(C) \cong \mathcal{O}_X(n)$ for some n (the degree of the hypersurface).

Since $i > 0$ we get that

$$H^i(X, \omega_X \otimes \mathcal{O}_X(n)) = 0$$

for all sufficiently high n as ω_X is coherent, by Theorem 3.34. This statement is independent of C so, in fact, we have to choose C so that the degree of the hypersurface is so high that the above equation holds. Then, by taking the long exact sequence corresponding to 4.2 where the cohomologies of the middle term vanish, we get $H^i(C, \omega_{C/\Bbbk}) \cong H^i(X, \omega_{X/\Bbbk})$ using that $H^i(C, \omega_{C/\Bbbk}) \cong H^i(X, i_*\omega_{C/\Bbbk})$.

Similar Bertini argument can help to find the pairing. □

4.2 Riemann-Roch for surfaces

Setup: From now on, $\Bbbk$ is an algebraically closed field and $X/\Bbbk$ is a smooth projective surface (i.e. integral scheme).

Recall: There exists a $\mathbb{Z}$-bilinear intersection pairing

$$(.,.) \; : \; \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$$

such that if $C, D \hookrightarrow X$ are irreducible curves, then $(C, D) := \deg_\Bbbk(i_{C*}D)$ where $i_C : C \hookrightarrow X$. One can check that it is the same as

$$\sum \dim_\Bbbk \mathcal{O}_{X,p}/(f_p, g_p)$$

where f_p and g_p are local equation of C and D. It also shows symmetry. We have seen that if $D = \mathrm{div}(f)$ then $(D, D') = 0$ for all$D' \in \mathrm{Div}(X)$.

Lemma 4.29. (Translation of the above for invertible sheaves) *Let $C, D \hookrightarrow X$ be irreducible curves. Then*

$$(C, D) = \deg_\Bbbk(i_{C*}\mathcal{L}(D))$$

Theorem 4.30. (Adjunction formula) *If $C \subseteq X$ is a smooth curve. Then*

$$\mathrm{genus}(C) = \frac{1}{2}(C, C + K) + 1$$

where $\mathcal{L}(K) = \omega_{X/\Bbbk}$.

Proof. Let us denote $g := \mathrm{genus}(C)$. By Riemann-Roch, we have

$$2g - 2 = \deg_\Bbbk \omega_{X/\Bbbk} =$$

which is the same as

$$= \deg_\Bbbk\big(i_C^*\omega_{X/\Bbbk} \otimes \mathcal{L}(C)\big) =$$

By the Lemma, it is just $(C, C + K)$. □

Theorem 4.31. (Riemann-Roch for surfaces) *Let $D \in \mathrm{Div}(X)$. Then*

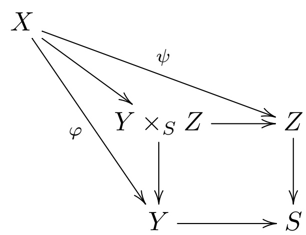
$$\chi\big(\mathcal{L}(D)\big) - \chi(\mathcal{O}_X) = \frac{1}{2}(D, D - K)$$

Lemma 4.32. (Enriques-Severi-Zariski) *Let X be a projective scheme over a field $\Bbbk$ and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L} \otimes \mathcal{O}_X(m)$ is very ample for m sufficiently large.*

Lemma 4.33. *Let $\varphi : X \to Y$ be a closed immersion of separated schemes over a base S and $\psi : X \to Z$ another morphism over S. Then*

$$\varphi \times \psi \ : \ X \to Y \times_S Z$$

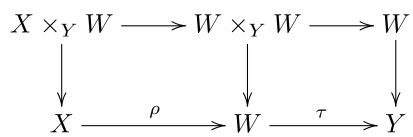
is a closed immersion such that the following diagram commutes:



Proof of Lemma 4.33. Recall that if $X \hookrightarrow Y$ is a closed immersion (over S) then for all $V \to S$ we have $X \times_S V \hookrightarrow Y \times_S V$ is a closed immersion (for affine ones, this follows by the right exactness of tensor product).

For $W := Y \times_S Z$ we have to prove that $X \xrightarrow{\rho} W \xrightarrow{\tau} Y$ where X, W and Y are separated over S and $\tau \circ \rho$ is a closed immersion then ρ is also a closed immersion. First, note that ρ factors as $X \xrightarrow{\Gamma_\rho} X \times_Y W \xrightarrow{p_2} W$. By separatedness, we get that Γ_ρ is a closed immersion. Then, as the composite of two closed immersions is a closed immersion, the statement follows by proving that p_2 is a closed immersion.

The proof depends on the commutativity of the following diagram



where the composition of the first row is p_2 and the composition in the second row is $\tau \circ \rho$ which is a closed immersion. Hence, the statement follows by the base change property of $\tau \circ \rho$. □

Proof of Lemma 4.32. Let $p \in X$ be a closed point, $\mathcal{I}_p$ be the ideal sheaf of functions vanishing at p. Then we get a short exact sequence

$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to \kappa(p) \to 0$$

Tensor it with $\mathcal{L}(m)$ obtaining

$$0 \to \mathcal{L} \otimes \mathcal{I}_p(m) \to \mathcal{L}(m) \to \mathcal{L} \otimes \kappa(p)(m) \to 0$$

By Serre's theorem 3.34, we get that $H^1(X, \mathcal{I}_p \otimes \mathcal{L}(m)) = 0$ for all sufficiently large m. For such an m we get that $H^0(X, \mathcal{L}(m)) \to H^0(X, \mathcal{L}(M) \otimes \kappa(p))$ is surjective, by the long exact sequence. By this, by an already used argument (involving Nakayama around p and taking finite covering) we get that $\mathcal{L}(m)$ is generated by global sections.

It means that $\mathcal{L}(m)$ gives rise to a map (that is not necessarily a closed immersion) $\varphi_{\mathcal{L}(m)} : X \to \mathbb{P}^N$ where N is the number of generating global sections. To apply the sublemma, we first give another morphism that is surely a closed immersion, namely, consider the d-uple embedding $\varphi_{\mathcal{O}(d)} : X \hookrightarrow \mathbb{P}^M$ where M is the number of monomials of degree d. Now, take the product of these two, giving

$$X \overset{(\varphi_{\mathcal{L}(m)}, \varphi_{\mathcal{O}(d)})}{\longrightarrow} \mathbb{P}^N \times \mathbb{P}^N \overset{\text{Segre}}{\longrightarrow} \mathbb{P}^{NM+N+M}$$

By the sublemma 4.33, $(\varphi_{\mathcal{L}(m)}, \varphi_{\mathcal{O}(d)})$ is a closed immersion. One can check that the composition is $\varphi_{\mathcal{L}(m) \otimes \mathcal{O}(d)} = \varphi_{\mathcal{L}(m+d)}$. The statement follows. □

Corollary 4.34. *Assume that X is a smooth projective surface, $D \in \mathrm{Div}(X)$. Then there exist smooth curves $C, C' \hookrightarrow X$, $[C - C'] = [D] \in \mathrm{Pic}(X)$.*

Remark 4.35. We will apply this corollary to change our arbitrary divisor in Theorem 4.31 to the difference of two curves. Also, note that the corollary is true in any dimension.

Proof. By Lemma 4.32, for all sufficiently large m, $\mathcal{L}(D)(m) \cong i^*\mathcal{O}(1)$ for some $i : X \hookrightarrow \mathbb{P}^N$. By Bertini's theorem, $i^*\mathcal{O}(1) \cong \mathcal{L}(C)$ for some smooth curve $C \hookrightarrow X$. Similarly, $i^*\mathcal{O}(m) \cong \mathcal{L}(C')$ for some smooth curve $C' \hookrightarrow X$. Then, we have $\mathcal{L}(D) \otimes \mathcal{L}(C') \cong \mathcal{L}(C)$ hence $\mathcal{L}(D) \cong \mathcal{L}(C - C')$. □

Proof of Theorem 4.31. Write $\mathcal{L}(D) \cong \mathcal{L}(C - C')$ as in the corollary. Then, we have two exact sequences

$$0 \to \mathcal{L}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0 \qquad \text{and} \qquad 0 \to \mathcal{L}(-C') \to \mathcal{O}_X \to \mathcal{O}_{C'} \to 0$$

Tensor them with $\mathcal{L}(C)$ giving

$$0 \to \mathcal{O}_X \to \mathcal{L}(C) \to \mathcal{O}_C \otimes \mathcal{L}(C) \to 0 \qquad \text{and} \qquad 0 \to \mathcal{L}(C - C') \to \mathcal{L}(C) \to \mathcal{O}_{C'} \otimes \mathcal{L}(C) \to 0$$

Then – using that χ is additive on short exact sequences – we get that

$$\chi(\mathcal{L}(C - C')) - \chi(\mathcal{O}_X) = \chi(\mathcal{L}(C) \otimes \mathcal{O}_C) - \chi(\mathcal{L}(C) \otimes \mathcal{O}_{C'})$$

Apply Riemann-Roch on C and C':

$$\chi(\mathcal{L}(C) \otimes \mathcal{O}_C) - \chi(\mathcal{O}_C) = \deg_{\Bbbk}(\mathcal{L}(C) \otimes \mathcal{O}_C) = (C, C)$$

$$\chi(\mathcal{L}(C) \otimes \mathcal{O}_{C'}) - \chi(\mathcal{O}_{C'}) = \deg_{\Bbbk}(\mathcal{L}(C) \otimes \mathcal{O}_{C'}) = (C, C')$$

Now, we are using the smoothness of C and C' that is required in the adjunction formula. It gives

$$-\chi(\mathcal{O}_C) := -\dim_{\Bbbk} H^0(C, \mathcal{O}_C) + \dim_{\Bbbk} H^1(C, \mathcal{O}_C) = -1 + g_c = \frac{1}{2}(C, C + K)$$

Similarly, we have $-\chi(\mathcal{O}_{C'}) = \frac{1}{2}(C', C' + K)$. Putting together all the equations give the statement. Namely,

$$\chi(\mathcal{L}(D)) - \chi(\mathcal{O}_X) = \chi(\mathcal{L}(C - C')) - \chi(\mathcal{O}_X) = \chi(\mathcal{L}(C) \otimes \mathcal{O}_C) - \chi(\mathcal{L}(C) \otimes \mathcal{O}_{C'})$$

$$= (C, C) - (C, C') + \frac{1}{2}(C', C' + K) - \frac{1}{2}(C, C + K) = \frac{1}{2}(D, D - K)$$

giving the statement. □

Remark 4.36. Serre duality will tell that $H^2(X, \mathcal{L}(D))$ is dual to $H^0(X, \mathcal{L}(K - D))$.

Corollary 4.37. $\dim_{\Bbbk} H^0(X, \mathcal{L}(D)) + \dim_{\Bbbk} H^0(X, \mathcal{L}(K - D)) \geq \frac{1}{2}(D, D - K) + \chi(\mathcal{O}_X)$. *The error term in this inequality is the H^1 term.*

Theorem 4.38. (Hodge index theorem) *Let X be a smooth surface, H a smooth hyperplane section and $D \in \mathrm{Div}(X)$. If $(D, H) = 0$ then $(D, D) \leq 0$.*

Proof. Assume that $(D, D) > 0$. It is enough to show that under this assumption $H^0(X, \mathcal{L}(mD)) \neq 0$ or $H^0(X, \mathcal{L}(-mD)) \neq 0$ for all sufficiently large m. This implies the theorem: if $0 \neq f \in H^0(X, \mathcal{L}(mD))$ then $mD + \mathrm{div}(f)$ is an effective divisor, by definition. However, in this case, $0 < (mD + \mathrm{div}(f), H)$ as an effective divisor corresponds to a sum of curves with positive coefficients and they all intersect H. We also have $(mD + \mathrm{div}(f), H) = (mD, H) = m(D, H)$ and we assumed that the latter is zero, and that is a contradiction. If $H^0(X, \mathcal{L}(-mD)) \neq 0$ then we get $(D, H) < 0$ and the same contradiction.

By Corollary 4.37, the sum $\dim_{\Bbbk} H^0(X, \mathcal{L}(mD)) + \dim_{\Bbbk} H^0(X, \mathcal{L}(K - mD))$ will converge to infinity as it is at least

$$\frac{1}{2}(mD, mD - K) + \chi(\mathcal{O}_X) = \frac{m^2}{2}(D, D) - \frac{m}{2}(D, K) + \chi(\mathcal{O}_X)$$

Similarly, for $-mD$. Assume that $H^0(X, \mathcal{L}(mD)) = H^0(X, \mathcal{L}(-mD)) = 0$. Then we get that $\dim_{\Bbbk} H^0(X, \mathcal{L}(K - mD))$ and $\dim_{\Bbbk} H^0(X, \mathcal{L}(K + mD))$ tends to infinity as $m \to \infty$. Generally, if $D_1, D_2 \in \mathrm{Div}(X)$ then $H^0(X, \mathcal{L}(D_1)) \neq 0$ and $\dim H^0(X, \mathcal{L}(D_1 + D_2)) \geq \dim H^0(X, \mathcal{L}(D_2))$ hence – by applying it to $D_1 = K - mD$ and $D_2 = K + mD$, we get $\dim H^0(X, \mathcal{L}(2K)) \geq \dim H^0(X, \mathcal{L}(K - mD))$ which goes to infinity. This is a contradiction. □

Remark 4.39. The name index theorem refers to the following: Consider the subgroup

$$N(X) := \{D \in \mathrm{Div}(X) \mid (D, D') = 0\ \forall D' \in \mathrm{Div}(X)\} \subseteq \mathrm{Div}(X)$$

the so called *numerically trivial divisors*. Take $\mathrm{Num}(X) := \mathrm{Div}(X)/N(X)$.

Fact 4.40. (Severi) $\mathrm{Num}(X)$ *is a finitely generated abelian group.*

So, the intersection pairing induces a non-degenerate $\mathbb{R}$-bilinear pairing on $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Its signature is an invariant of the basis. The Hodge index theorem says exactly that the signature is $(+1, -1, \dots, -1)$ i.e. that the index of the form is $n - 1$ where n is the number of generators of $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

ELEVENTH LECTURE, 31TH OF MARCH

5 Moduli spaces

Sources: (For Hilbert schemes, stacks, moduli spaces)

- Nitsure, Hilbert Schemes, in Fundamental Algebraic Geometry (Fantecki, et al.) (~30 pages intro)
- Mumford, Lectures on Curves on an algebraic surface (original reference)
- M. Olsson, Introduction to (algebraic?) stacks (new book)
- D. Edidin, Notes on moduli spaces of curves (on arxiv)
- Handbook of Moduli
- Mumford, Suominen, Introduction to Moduli, Conference in Oslo, 1970 (elementary examples)
- Mumford, Geometric Invariant Theory (hard)
- Le Potier, Lectures on vector bundles (restricted cases)

Reminder: Let S be a base scheme, and X a scheme over S. Then the functor of points of X is

$$\mathrm{Hom}_{\mathrm{Sch}/S}(.,X) \ : \ \mathrm{Sch}/S \to \mathrm{Sets}$$

The origin of the name comes from the fact that $\Bbbk$-points of a variety over $\Bbbk$ correspond to maps $\mathrm{Spec}\Bbbk \to T$ over $\mathrm{Spec}\Bbbk$. By Yoneda's lemma, we do not lose any information by considering functor of points, since the functor uniquely determines X if it exists.

Example 5.1. Consider the functor associating to a scheme T over S all the quotients of $\mathcal{O}_T^{n+1}$ onto an invertible sheaf on T (up to isomorphism of sheaves). This is the functor of point of $\mathbb{P}^n_S$. Indeed, recall that there exists a bijective correspondence between

$$\mathrm{Hom}_{\mathrm{Sch}/S}(T, \mathbb{P}^n_S) \longleftrightarrow \{\text{invertible sheaves on } T \text{generated by } n+1 \text{global sections}\}$$

given by $\left(\varphi : T \to \mathbb{P}^n_S\right) \mapsto \varphi^*\mathcal{O}(1)$ and the surjections is given by mapping each coordinate of $\mathcal{O}_T^{n+1}$ onto generators of the invertible sheaf.

Example 5.2. More generally, consider the functor that associates to a scheme T over S all the quotients of $\mathcal{O}_T^{n+1}$ onto an locally free sheaf on T of rank r (up to isomorphism of sheaves). This is the functor of point of $\mathrm{Gr}(n,r)_S$.

Example 5.3. A basic moduli problem: What about the functor $\mathcal{M}_g$ that associates to a scheme T over S all maps $C \to T$ that are surjective, flat and the fibers are smooth projective curves of genus g (up to isomorphism of maps over S). In particular, if $T = \mathrm{Spec}\mathbb{k}$ for $\mathbb{k} = \overline{\mathbb{k}}$ then $\mathcal{M}_g$ is the isomorphism classes of smooth projective curves over $\mathbb{k}$ of genus g.

Suppose that $\mathcal{M}_g$ is representable by a scheme $M_g \to S$ i.e. $\mathcal{M}_g \cong \mathrm{Hom}_{\mathrm{Sch}/S}(., M_g)$. For $T = M_g$, id_{M_g} will correspond to an element in $\mathcal{M}_g(M_g)$ which is a curve $C_g \to M_g$ called the universal curve. Then if $(C \to T) \in \mathcal{M}_g(T)$ then C is the pullback of $C_g \to M_g$ i.e. there exists a map $T \to M_g$ such that in the pullback diagram

$$\begin{array}{ccc} C' & \longrightarrow & C_g \\ \downarrow & & \downarrow \\ T & \longrightarrow & M_g \end{array}$$

we have $C' \cong C$. Unfortunately, this is not true (hence this functor is not representable, M_g does not exist as a scheme).

Suppose that P_1 and P_2 are different points $\mathrm{Spec}\overline{\mathbb{k}} \to T$. Then the two maps $\mathrm{Spec}\overline{\mathbb{k}} \to T \to M_g$ give two curves C_1 and C_2 over $\overline{\mathbb{k}}$ but they arise as fibers in a family $C \to T$ by pulling back $C_g \to M_g$ onto T. If C_1 and C_2 would be isomorphic curves, then they should correspond to the same point $\mathrm{Spec}\overline{\mathbb{k}} \to M_g$. Similarly, assume that $C \to T$ is a family of curves with all fibers isomorphic, then it must correspond to a constant morphism $T \to M_g$, so $C \to T$ is isomorphic to $C_1 \times_S T \to T$. However, there exist families with isomorphic fibers that are not direct products:

Example 5.4.

1. Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $(x_0, x_1, x_2) \mapsto (x_0, x_1)$ which is defined everywhere except $(0,0,1)$. However, if we blow it up at $(0,0,1)$ then it will extend to the closure of the blowup with fibers $\mathbb{P}^1$ but $V \cong \mathbb{P}^1 \times \mathbb{P}^1$.

2. Let C be a curve over $\overline{\mathbb{k}}$ with $G = \mathrm{Aut}(C)$ finite e.g. when $g \geq 2$. Take another X with a free action of G and consider $C \times X$ with the product action of G. Then by defining $Y := (C \times X)/G$ we get a family of curves $C \times X \to Y$ with fibers C but it is not a direct product.

Remark 5.5. Let G be a finite group. We may define X/G as follows: If $X = \mathrm{Spec}A$ then $X/G := \mathrm{Spec}A^G$ corresponding to orbits of G. If we have an affine open covering $X = \cup_i \mathrm{Spec}A_i$ such that every G-orbit is contained in one of the $\mathrm{Spec}A_i$'s, then we can patch X/G together.

Remark 5.6. The second example is very important since if curves had no automorphisms then we could not construct counterexamples so the moduli space "would" exist. So a general goal in moduli space theory is to kill these automorphisms by attaching extra data to curves that breakes the automorphisms.

Solutions:

1. M_g exists but not as a scheme but as an algebraic stack (or orbifold).
2. Coarse moduli spaces (proposed by Mumford): Although M_g does not exists but there exists an $\tilde{M}_g$ such that
   - $\mathcal{M}_g(\mathrm{Spec}\overline{\mathbb{k}}) \cong \tilde{M}_g(\mathrm{Spec}\overline{\mathbb{k}})$ for any point $\mathrm{Spec}\overline{\mathbb{k}} \to S$ and

- for every morphism of functors $\mathcal{M}_g \to \mathrm{Hom}_{\mathrm{Sch}/S}(., M)$ where $M \to S$ is a scheme over S factorize through $\mathrm{Hom}_{\mathrm{Sch}/S}(., \tilde{M}_g)$

$$\begin{array}{ccc} \mathcal{M}_g & \longrightarrow & \mathrm{Hom}_{\mathrm{Sch}/S}(., \tilde{M}_g) \\ & \searrow & \downarrow \\ & & \mathrm{Hom}_{\mathrm{Sch}/S}(., M) \end{array}$$

Theorem 5.7. (Mumford) *$\tilde{M}_g$ exists.*

Proposition 5.8. *If $g \geq 2$ then every smooth curve of genus g embeds in $\mathbb{P}^{5g-6}$. In this case, $\big(\Omega^1_{C/\Bbbk}\big)^{\otimes 3}$ is very ample and $\deg(\Omega^1_{C/\Bbbk})^{\otimes 3} = 6g - 6$ by Riemann-Roch and the addivity of degree on tensor. If $g \geq 2$ then $6g - 6 > 2g$ hence – by previous semester – we have*

$$\dim H^0(C, (\Omega^1)^{\otimes 3}) = \deg(\Omega^1)^{\otimes 3} - g + 1 = (6g - 6) - g + 1 = 5g - 5$$

so it embeds into $\mathbb{P}^{5g-6}$.

Corollary 5.9. *If we want to classify curves of genus $g \geq 2$ then it may be a better idea to classify the curves of degree $6g - 6$ in $\mathbb{P}^{5g-6}$ and then to find out which are isomorphic among them.*

Problem 5.10. We want to classify closed subschemes in $\mathbb{P}^N$ with fixed numberical invariants (like the degree of a curve).

Definition 5.11. Flatness: Let A be a ring. An A-module B is flat if for any an exact sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

the tensored sequence

$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$$

is exact (i.e. $\otimes B$ is an exact functor). This is a generalization to quasi-coherence in the case of sheaves. A morphism $\varphi : T \to S$ of schemes is flat if $\varphi_* \mathcal{O}_T$ is flat over $\mathcal{O}_S$.

Fact 5.12. *In a flat family, the dimension of fibers is constant.*

Exercise 5.13. Let $\Bbbk = \overline{\Bbbk}$, $A = \Bbbk[t]$ and $B = \Bbbk[t, x]/(tx - x)$. Then the fibers of $\mathrm{Spec} B \to \mathrm{Spec} A \cong \mathbb{A}^1$: over $t = 1$ the fiber is $\mathrm{Spec}\Bbbk[x]$ and over $t \neq 1$ the fibers are $\mathrm{Spec}\Bbbk$. This phenomenon is possible since B is not flat over A. Indeed, $t \in A$ is a zero divisor in B hence $(0 \to A \xrightarrow{\cdot t} A \to A/tA) \otimes B$ is not exact.

Definition 5.14. We want to consider the functor Hilb_N that associated to a scheme T over S the closed subschemes $Z \hookrightarrow T \times_S \mathbb{P}^N_S$ such that $p_1 : Z \to T$ is flat, up to isomorphism of embeddings.

There are other invariants of fibres that are constant on flat families. Such an invariant is the Hilbert polynomial.

Proposition 5.15. *Let $\mathcal{F}$ be a coherent sheaf on projective space $\mathbb{P}^n$. Consider $\chi\big(\mathcal{F}(m)\big)$ which makes sense by Serre's finiteness theorem. This is a polynomial of degree at most n with $\mathbb{Q}$-coefficients in m for m sufficiently large.*

Sketch of the proof. By Serre's theorem, we know that $H^i(\mathbb{P}^n, \mathcal{F}(m)) = 0$ for m sufficiently large. Hence, for m sufficiently large,

$$\chi(\mathcal{F}(m)) = \dim H^0(\mathbb{P}^n, \mathcal{F}(m))$$

If $\mathcal{F} = \mathcal{O}(d)$ then this is $\dim H^0(\mathbb{P}^n, \mathcal{F}(d + m)) = \binom{n+d+m}{n}$ which is a polynomial in m with rational polynomials. Otherwise, there exists a surjection $\oplus_{i=1}^r \mathcal{O}(d_i) \twoheadrightarrow \mathcal{F}$, in fact, we can have a whole free resolution. This resolution is finite, this is an appropriate graded version of the Hilbert syzygy theorem. Since the theorem is true for $\mathcal{O}(d_i)$'s, we get the statement for $\mathcal{F}$ by the additivity of χ on exact sequences. □

Definition 5.16. The above polynomial $p_{\mathcal{F}} \in \mathbb{Q}[m]$ is the Hilbert polynomial of $\mathcal{F}$. If $\mathcal{F} = i_*\mathcal{O}_Z$ for a closed subscheme $i : Z \hookrightarrow \mathbb{R}^n$, $p_Z := p_{\mathcal{F}}$ is the Hilbert polynomial of Z.

Fact 5.17. $\deg p_Z = \dim Z$ *and if* $Z \hookrightarrow \mathbb{P}^n$ *is a curve of degree* d *then* d *is the leading coefficient of* p_Z*, i.e.* $p_Z = dm + c$.

Proposition 5.18. *If* T *is integral and Noetherian then for a closed subscheme* $Z \hookrightarrow \mathbb{P}^N_T$ *and for a flat morphism* $Z \to T$*,* p_{Z_t} *is independent of* t *where* Z_t *is the fiber over* $t \in T$.

Sketch of the proof. Replace T by $\mathrm{Spec}\mathcal{O}_{T,t}$ i.e. we reduce to the case where $T = \mathrm{Spec}A$ where A is Noetherian and local. We can do this because the scheme is integral hence it has a generic fiber so it is enough to prove that the Hilbert polynomial of the fiber over each point is the same as over the generic point. Hence, it is enough to prove this for the generic point of $\mathcal{O}_{T,t}$ and for the closed point $(t) \lhd \mathcal{O}_{T,t}$.

We have an exact sequence

$$0 \to H^0(\mathbb{P}^n, \mathcal{O}_Z(m)) \to \bigoplus H^0(D_+(x_i), \mathcal{O}_Z(m)) \to H^0(D_+(x_i) \cap D_+(x_i), \mathcal{O}_Z(m))$$

where the second and third term is flat over A, as the structure sheaf is flat. By generalities, it follows that then the first term is also flat. But it is also finitely generated over A by Serre's theorem. However, a finitely generated flat module over a commutative ring is also free. Let $r_m := \mathrm{rk}H^0(\mathbb{P}^n, \mathcal{O}_Z(m))$. Since it is a free module, its rank over the closed point is the same as over the generic point. One can show that $\dim H^0(\mathbb{P}^n, \mathcal{O}_{Z_t}(m)) = r_m$ for $t \in T$. □

Definition 5.19. So we modify our functor: We want to consider the functor $Hilb^p_N$ that associated to a scheme T over S the closed subschemes $Z \hookrightarrow T \times_S \mathbb{P}^N_S$ such that $p_1 : Z \to T$ is flat, and the Hilbert polynomial of Z_t is a fixed p for all $t \in T$ (up to isomorphism of embedding). So we get $Hilb_N = \cup_p Hilb^p_N$ (union means disjoint union in the category Sets.

Theorem 5.20. (Grothendieck) $Hilb^p_N$ *is representable by a scheme* $\mathrm{Hilb}^p_N \to S$.

Example 5.21.

1. If $p = m + 1$ then Z is a family of lines by the fact that the main coefficient is the degree. Hence, $\mathrm{Hilb}^p_N = \mathbb{P}^n$.
2. If $p = \binom{N+m}{N} - \binom{N-d+m}{m}$ then Z is a family of degree d hypersurfaces. Hence, $\mathrm{Hilb}^p_N = \mathbb{P}^{\binom{N+d}{d}-1}$.
3. If $p = \binom{r+m}{m}$ then $\mathrm{Hilb}^p_N = \mathrm{Gr}(r + 1, m + 1)$.
4. Curves embedded into $\mathbb{P}^{5g-6}$ by $(\Omega^1_{X/\Bbbk})^{\otimes 3}$ are contained in Hilb^p_N for $p = (6g - 6)m + (g - 1)$. To get all the curves that come from the canonical embedding of a curve of genus $g \geq 2$ via $\Omega^1_{X/\Bbbk}$, we have to restrict ourselves to a locally closed subscheme where $\mathcal{O}(1)|_X \cong (\Omega^1_{X/\Bbbk})^{\otimes 3}$.

Idea of the proof of Grothendieck's theorem: Consider the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^n_T} \to i_*\mathcal{O}_Z \to 0$$

where Z is a closed subscheme of $\mathbb{P}^n_T$ over T.

Step 1: (Castelnouvo, Mumford) There exists a constant $N \geq 0$ depending only on the coefficients of $P := P_{Z_t}$ $(t \in T)$ such that $\mathcal{O}_{Z_t}(m)$ and $K|_{Z_t}(m)$ are generated by global sections and have trivial higher cohomology, for all $m \geq N$.

So if $\pi : \mathbb{P}^n_T \to T$ is the projection then we have an exact sequence $\pi_*\mathcal{O}_{\mathbb{P}^n_T(m)} \to \pi_*\mathcal{O}_Z(m) \to (R^1\pi_*)\kappa(m)$. The last term for m sufficiently large, the middle term is locally free by Serre's theorem and flatness, and the first term is a quotient of $\mathcal{O}_T^{n+1}$ as it is globally generated. So it will define a T-point in a Grassmannian for appropriate parameteres, as Grassmannian classifies locally free quotients.

Step 2: This map is injective and the image is a locally closed subscheme. This is done via the method of flattening stratifications.

Back to M_g : Assume that $C \hookrightarrow \mathbb{P}^{5g-6}$ is a canonically embedded curve of genus g by $\left(\Omega^1_{C/\Bbbk}\right)^{\otimes 3}$. We may associate to C as point of $\mathrm{Hilb}^{(6g-6)m-g+1}_{5g-6}$. These points lie in a locally closed subscheme.

Fact 5.22. *If C and C' are embedded in $\mathbb{P}^{5g-6}$ via $(\Omega^1)^{\otimes 3}$ then they are isomorphic if and only if there exists an automorphism of $\mathbb{P}^{5g-6}$ inducing this isomorphism.*

$\tilde{M}_g$ is contructed as a quotient of H_g by the action of $PGL(5g-6)$. It exists as a schem (coarse moduli space) by geometric moduli space, or as a stack.

Last Lecture on the topics of GAGA, 13th of April

6 GAGA

Motivation: Riemann Existence theorem: Every compact Riemann surface is algebraic, i.e. it is isomorphic to a smooth projective complex curve with its "natural" analytic structure. Equivalently, there exists a non-constant meromorphic function on a compact Riemann surface. Indeed, a meromorphic function $f : X \to \mathbb{P}^1_{\mathbb{C}}$ defines a finite branched covering of $\mathbb{P}^1_{\mathbb{C}}$ which can be already embedded into $\mathbb{P}^n_{\mathbb{C}}$. Then, we may apply the following:

Theorem 6.1. *Chow's Theorem: Every closed complex analytic subvariety of $\mathbb{P}^n_{\mathbb{C}}$ is algebraic. Here, analytic subvariety means that it is the common zero locus of holomorphic functions.*

Example 6.2. Compactness is crucial here, e.g. $\mathbb{C}\setminus\{\text{finitely many points}\}$ is a non-algebraic Riemann surface.

Note that it can also be applied to graphs of functions between compact Riemann surfaces, hence every mapping is also algebraic. In fact, more is true:

Proposition 6.3. (GAGA) *Every coherent analytic sheaf on $\mathbb{P}^n_{\mathbb{C}}$is algebraic.*

By this, we can deduce Chow's theorem: Indeed, if we take some holomorhic functions then they generate an idael sheaf $\mathcal{I}$ of the analytic structure sheaf $\mathcal{O}^{\mathrm{an}}_{\mathbb{P}^n}$. If the statements in GAGA are true then it corresponds to an ideal sheaf in $\mathcal{O}_{\mathbb{P}^n}$ so we may take $\mathrm{Supp}(\mathcal{O}/\mathcal{I})$ which will be the (already algebraic) subvariety we started with.

What's with Riemann Existence theorem. Recall that if $\varphi : X \to Y$ is a finite morphism of schemes then $\varphi_*\mathcal{O}_X$ is finitely generated as an $\mathcal{O}_Y$-algebra. The analogous proposition is true for finite maps of analytic manifolds. By GAGA, they are equivalent. The case of $Y = \mathbb{P}^1_{\mathbb{C}}$ gives the Riemann Existence theorem.

Remark 6.4. Some parts of the statements can be saved in the p-adic setup.

6.1 Complex analytic spaces

Definition 6.5. The sheaf of holomorphic functions on $\mathbb{A}^n_{\mathbb{C}}$ is denoted by $\mathcal{O}^{\mathrm{an}}$. The pair $(\mathbb{A}^n_{\mathbb{C}}, \mathcal{O}^{\mathrm{an}})$ is a locally ringed space.

Analogously to the algebraic case, the functions $f_1, \dots, f_r \in \mathcal{O}^{\mathrm{an}}(\mathbb{A}^n)$ define a generated ideal sheaf $\mathcal{I}$ and we may take $\mathrm{Supp}(\mathcal{O}^{\mathrm{an}}/\mathcal{I}) =: Z \subseteq \mathbb{A}^n$ that will be (analytically) closed. It is again a complex analytic space by $\mathcal{O}^{\mathrm{an}}_Z := \left(\mathcal{O}^{\mathrm{an}}/I\right)|_Z$ called a closed analytic subspace of $\mathbb{A}^n$.

More generally, an analytic subspace of $\mathbb{A}^n$ is an open subset of a closed analytic subspace.

Definition 6.6. (Grothendieck) A complex analytic space is a locally ringed space $(Y, \mathcal{O}_Y)$ such that there exists a covering $\cup U_i = Y$ such that $(U_i, \mathcal{O}_Y|_{U_i})$ is isomorphic to an analytic subspace of $\mathbb{A}^n$ as a locally ringed space.

Remark 6.7. Serre also required the space to be T_2 and open subsets were not allowed.

Proposition 6.8. (Grothendieck) *Let X be a scheme of finite type over $\mathbb{C}$. There exists a complex analytic space X^{an} and a morphism of locally ringed spaces $\varepsilon : X^{\mathrm{an}} \to X$ such that for any morphism $Y \to X$ where Y is a complex analytic space, we can uniquely complete the following diagram:*

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & X^{\mathrm{an}} \\ & \searrow & \downarrow \varepsilon \\ & & X \end{array}$$

For the proof, see the notes on the webpage of the lecturer.

Definition 6.9. A coherent analytic sheaf on a complex analytic space $(Y, \mathcal{O}_Y)$ is an $\mathcal{O}_Y$-module $\mathcal{F}$ such that there exists an open covering $\cup U_i = Y$ such that for all i,

$$\mathcal{F}|_{U_i} \cong \operatorname{Coker}\big(\mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus s}\big)$$

Proposition 6.10. *If X is a finite type scheme oevr $\mathbb{C}$ and $\mathcal{F}$ is a coherent sheaf on X then*

$$\mathcal{F}^{\mathrm{an}} := \varepsilon^* \mathcal{F} = \varepsilon^{-1} \mathcal{F} \otimes_{\varepsilon^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$$

$\mathcal{F}^{\mathrm{an}}$ is also coherent (as an analytic sheaf).

Theorem 6.11. *Let X be a closed subscheme of $\mathbb{P}^N_{\mathbb{C}}$. The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$*

1. *induces an equivalence of categories.*
2. *induces an isomorphism $H^i(X, \mathcal{F}) \overset{\cong}{\to} H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ for all $i \geq 0$.*

Remark 6.12. Note that both are sheaf cohomologies that use only the (analytic) topology of X^{an}.

This applies to differential forms too. Assume that X is a smooth projective scheme. Then we get that

$$H^p(X, \Omega^q_{X/\mathbb{C}}) \overset{\cong}{\to} H^p(X^{\mathrm{an}}, \Omega^q_{X^{\mathrm{an}}/\mathbb{C}})$$

for all $p, q \geq 0$. Consequently,

$$H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X^{\mathrm{an}}) \overset{\text{by Hodge-theory}}{\cong} \bigoplus_{p+q=i} H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \Omega^q_{X^{\mathrm{an}}}) \overset{\cong}{\leftarrow} \bigoplus_{p+q=i} H^p(X, \Omega^q_X)$$

Note, however, that the most left isomorphism uses the Poincaré lemma for analytic spaces which is not true for algebraic spaces. Still the algebraic Hodge decomposition on the most right hand side is isomorphic to the others.

Proposition 6.13. *The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is exact.*

Remark 6.14. Serre invented the notion of flatness for this proof.

Proof. This property can be checked on stalks. However, $(\mathcal{F}^{\mathrm{an}})_p \cong \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X^{\mathrm{an}},p}$. Fact: $\mathcal{O}_{X^{\mathrm{an}},p}$ is flat, even faithfully flat over $\mathcal{O}_{X,p}$. Recall that an A-module is faithfully flat if it is flat and $M \otimes N = 0$ only if $N = 0$. Moreover, recall that if A is Noetherian local then its completion $\hat{A} := \lim A/M^i$ is faithfully flat over A.

Lemma 6.15. *Assume that $A \to B$ is a local homomorphism of Noetherian localy rings, inducing an isomorphism $\hat{A} \overset{\cong}{\to} \hat{B}$. Then B is faithfully flat over A.*

In our case, we have $\hat{\mathcal{O}}_{X,p} \cong \hat{\mathcal{O}}_{X^{\mathrm{an}},p}$ using the embedding of $\mathcal{O}_{X,p}$ into $\mathcal{O}_{X^{\mathrm{an}},p}$. Hence, we can apply the lemma and the proposition follows. □

Construction of $H^i(X,\mathcal{F}) \to H^i(X^{\mathrm{an}},\mathcal{F}^{\mathrm{an}})$: Take a injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$ (using the Zariski topology). We may apply ε^* giving

$$\varepsilon^*\mathcal{F} \to \varepsilon^*\mathcal{I}^\bullet$$

which is again a resolution (i.e. it is exact) by the Proposition. However, they are not necessarily injective.

Lemma 6.16. *If $\mathcal{G} \to \mathcal{M}^\bullet$ is any resolution, $\mathcal{G} \to \mathcal{I}^\bullet$ is an injective resolution then we have a chain map $(\mathcal{G} \to \mathcal{M}^\bullet) \to (\mathcal{G} \to \mathcal{I}^\bullet)$ by injectivity.*

Now, we can apply this for an injective resolution $\mathcal{J}^\bullet$ of $\varepsilon^*\mathcal{F}$ with respect to the analytic topology. Then we get a map $\varepsilon^*\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ hence we have

$$I^\bullet \to \varepsilon_*\varepsilon^*\mathcal{I}^\bullet \to \varepsilon_*\mathcal{J}^\bullet$$

It gives a map between the cohomologies: $H^i(X,\mathcal{I}^\bullet) \to H^i(X,\varepsilon_*\mathcal{J}^\bullet) \cong H^i(X^{\mathrm{an}},\mathcal{J}^\bullet)$.

Theorem 6.17. (Cartan-Serre) *Let $\mathcal{F}$ be a coherent analytic sheaf on $\mathbb{P}^n_{\mathbb{C}}$. Then*

1. *$H^i(\mathbb{P}^n_{\mathbb{C}},\mathcal{F})$ is a finite dimensional $\mathbb{C}$-vector space for $i \geq 0$.*
2. *$H^i(\mathbb{P}^n_{\mathbb{C}},\mathcal{F}(m)) = 0$ for $i > 0$ and high enough $m > 0$.*

of Theorem REF, part 2. **Step_1:** Reduction to the case $X = \mathbb{P}^n_{\mathbb{C}}$. If $\mathcal{F}$ is a coherent sheaf on X and $i : X \to \mathbb{P}^n_{\mathbb{C}}$ is an embedding then we have

$$H^i(X,\mathcal{F}) \cong H^i(\mathbb{P}^N, i_*\mathcal{F})$$

which is proved in REF. But it is also true in the analytic case:

$$H^i(X^{\mathrm{an}},\mathcal{F}^{\mathrm{an}}) \cong H^i(\mathbb{P}^N, i_*\mathcal{F}^{\mathrm{an}})$$

because i_* is exact as i is a closed immersion.

Step_2: In the case of $\mathcal{F} = \mathcal{O}(m)$ we use induction on n. For $n = 0$ it is zero. For $H = \{x_0 = 0\} \subseteq \mathbb{P}^n$ we have an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

More generally, we have

$$0 \to \mathcal{O}(m-1) \to \mathcal{O}(m) \to \mathcal{O}_H(m) \to 0$$

by the projection formula REF. The same is true in the analytic case. Hence, we get two long exac sequences

$$\cdots \to H^i\big(\mathcal{O}(m-1)\big) \to H^i\big(\mathcal{O}(m)\big) \to H^i\big(\mathcal{O}_H(m)\big) \to H^{i+1}\big(\mathcal{O}(m-1)\big) \to \dots$$

$$\cdots \to H^i\big(\mathcal{O}^{\mathrm{an}}(m-1)\big) \to H^i\big(\mathcal{O}^{\mathrm{an}}(m)\big) \to H^i\big(\mathcal{O}_H^{\mathrm{an}}(m)\big) \to H^{i+1}\big(\mathcal{O}^{\mathrm{an}}(m-1)\big) \to \dots$$

but we have isomorphisms on the middle two by induction (on n and m) hence, by the 5-lemma we get the isomorphism for higher cohomologies. Still, we have to check how this inductino starts, i.e. $i = 0$ or $m = 0$ case. In the case, $i = 0$ we have

$$0 \to H^0\big(\mathcal{O}(m-1)\big) \to H^0\big(\mathcal{O}(m)\big) \to H^0\big(\mathcal{O}_H(m)\big) \to H^1\big(\mathcal{O}(m-1)\big) \to \dots$$

$$0 \to H^0\big(\mathcal{O}^{\mathrm{an}}(m-1)\big) \to H^0\big(\mathcal{O}^{\mathrm{an}}(m)\big) \to H^0\big(\mathcal{O}_H^{\mathrm{an}}(m)\big) \to H^1\big(\mathcal{O}^{\mathrm{an}}(m-1)\big) \to \dots$$

By induction, we have an isomorphism at $H^0\big(\mathcal{O}_H(m)\big)$ by induction on n. If $m = 0$ then we get $H^0(\mathbb{P}^n, \mathcal{O}) = H^0(\mathbb{P}^n, \mathcal{O}^{\mathrm{an}}) \cong \mathbb{C}$ by hand so this case is ok. For $m < 0$ we can use a descending induction.

Step_3: In the general case, we can take a surjection $\bigoplus_i \mathcal{O}(m_i) \to \mathcal{F}$. Then one can take the kernel of this surjection, use the long exact sequence of cohomologies. By Serre-Cartan and Serre's vanishing theorem it is enough. In the analytic case, an analogous proof works where instead of the vanishing theorem, one has to use a theorem of Cartan.

□

Proof of Theorem REF, part 1. Using flatness, one can prove the following:

Lemma 6.18. *$\mathcal{H}om(\mathcal{F},\mathcal{G})^{\mathrm{an}} \overset{\cong}{\to} \mathcal{H}om(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$ via ε_*.*

Now, we can consider the zeroth cohomology where we get isomorphism by the first part

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) = H^0(X, \mathcal{H}om(\mathcal{F},\mathcal{G})) \cong H^0(X^{\mathrm{an}}, \mathcal{H}om(\mathcal{F},\mathcal{G})^{\mathrm{an}}) \overset{\text{Lemma}}{\cong} \mathrm{Hom}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$$

Now, we have to prove that for any coherent analytic sheaf $\mathcal{G}$ on X^{an} there exists a coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}^{\mathrm{an}} \cong \mathcal{G}$.

Step_1: Reduction to $X = \mathbb{P}^n_{\mathbb{C}}$.

Step_2: There is a surjection such that $\bigoplus_i \mathcal{O}^{\mathrm{an}}(m_i) \to \mathcal{G}$ by the same proof as in the algebraic case with kernel $\mathcal{K}$. Similarly we can take its cover $\oplus_j \mathcal{O}^{\mathrm{an}}(m'_j) \to \mathcal{K}$. Then the modules of the form $\mathcal{O}^{\mathrm{an}}(m)$ come from the algebraic sheaves $\mathcal{O}(m)$. Moreover, we haveí

$$\mathrm{Hom}\big(\oplus_j \mathcal{O}^{\mathrm{an}}(m'_j), \bigoplus_i \mathcal{O}^{\mathrm{an}}(m_i)\big) \cong \mathrm{Hom}\big(\oplus_j \mathcal{O}(m'_j), \bigoplus_i \mathcal{O}(m_i)\big)$$

by the previous part of fully faithfulness. Hence, no matter which map defines $\mathcal{G}$ as a cokernel, it is an algebraic morphism hence its cokernel is isomorphic to $\mathcal{G}$ by faithfully flatness.

□