Basic Algebraic Geometry

September 21, 2016

Remark. This is the live-texed notes of the Basic Algebraic Geometry course held by Tamás Szamuely in the fall of 2015. I did not have the time to check it yet, so it is full of mistakes, typos and missing parts that I did not understand on the lecture.

First Lecture, 22nd of September

Prerequisites:

- Atiyah - MacDonald: Introduction to Commutative Algebra
- S. Lang: Algebra (just as a reference book for algebraic results)

Literature:

- I. Shafarevich: Basic Algebraic Geometry I
- R. Hartshorne: Algebraic Geometry, chapter 1
- Szamuely: Lectures notes on Linear Algebraic Groups (see homepage)

1 Affine varieties

Assumptions: In this course, $\Bbbk$ will be an algebraically closed field (unless explicitly stated otherwise) and the notation $\mathbb{A}^n_{\Bbbk} = \mathbb{A}^n_{\Bbbk} := \{(a_1, \dots, a_n) \in \Bbbk^n\}$ will be used.

Definition 1.1. A subset in $\mathbb{A}^n$ is called an *affine closed set* if there exists an ideal $I \lhd \Bbbk[x_1, \dots, x_n]$ such that this subset is the zero locus of I, namely if it is of the form:

$$V(I) := \{p \in \mathbb{A}^n \mid f(p) = 0, \ \forall f \in I\}$$

Remark 1.2. We know that every ideal of the ring $\Bbbk[x_1, \dots, x_n]$ is finitely generated: This is the Hilbert Basis Theorem. This means that if $I = (f_1, \dots, f_n)$ then $V(I) = V\big((f_1, \dots, f_n)\big)$ i.e. the points $p \in \mathbb{A}^n$ such that $f_i(p) = 0$ for all $i \leq n$.

Definition 1.3. Given an affine closed set $X \subseteq \mathbb{A}^n$ we can define its *vanishing ideal* as

$$I(X) := \{f \in \Bbbk[x_1, \dots, x_n] \mid f(p) = 0, \ \forall p \in X\}$$

which is obviously an ideal of the polynomial ring (checking is straightforward).

One may have the guess that the above two operations are inverses of each other but this is not always true. What is true in general, on one hand, is that $I \subseteq I(V(I))$ for any ideal I.

Example 1.4. Consider the ring $\Bbbk[x]$ and its ideal (x^2). Then $V\big((x^2)\big) = \{0\}$ but $I\Big(V\big((x^2)\big)\Big) = (x) \neq (x^2)$.

One necessary condition for being $I(X)$ for some affine closed set X is that

$$\sqrt{I(X)} := \big\{f \in \Bbbk[x_1, \dots, x_n] \mid \exists n \in \mathbb{N}: \ f^n \in I(X)\big\} = I(X)$$

i.e. it is necessarily a radical ideal which property is not satisfied by all the ideals. It is a theorem that it is also sufficient for being an $I(X)$:

Theorem 1.5. (Hilbert's Nullstellensatz, for proof see references) *For an ideal $I \lhd \Bbbk[x_1, \dots, x_n]$ we have $I(V(I)) = \sqrt{I}$.*

Corollary 1.6. *The map $I \mapsto V(I)$ induces an inclusion-reversing bijection between*

$$\big\{\textit{ideals } I \lhd \Bbbk[x_1, \dots, x_n] \textit{ such that } I = \sqrt{I}\big\} \longleftrightarrow \big\{\textit{closed sets in } \mathbb{A}^n\big\}$$

Remark 1.7. In the statement of the Nullstellensatz, it is crucial to have an algebraically closed field, it is used unavoidably in the proof.

Example 1.8.

1. If we take the non-algebraically closed field $\Bbbk = \mathbb{R}$ then $V(x^2 + y^2 + 1) = \emptyset$ so "the Nullstellensatz fails" here.
2. The affine closed sets in $\mathbb{A}^1$ are nothing but $\emptyset$, $\mathbb{A}^1$ and the finite sets. (Proof: they are determined by polynomials in one variable...)
3. The affine closed sets in $\mathbb{A}^2$ are $\emptyset$, $\mathbb{A}^2$, the finite sets, sets of the form $\big\{(x, y) \in \mathbb{A}^2 \mid f(x, y) = 0\big\}$ for some $f \in \Bbbk[x, y]$ and finite unions of these. (It is clear that these are affine closed sets but it is not yet obvious why there is nothing else.)

Lemma 1.9.

1. *For $I, J \in \Bbbk[x_1, \dots, x_n]$ we have $V(I \cdot J) = V(I) \cup V(J)$ where $I \cdot J := \big\{\sum_{i \in I} a_i b_i \mid |I| < \infty, \ a_i \in I, \ b_i \in J\big\}$.*
2. *For a set of ideals $\{I_\lambda \mid \lambda \in \Lambda\}$ where all $I_\lambda \lhd \Bbbk[x_1, \dots, x_n]$ we have*

$$V\big(\langle I_\lambda \mid \lambda \in \Lambda \rangle\big) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$$

Proof. $V(I \cdot J) \supseteq V(I) \cup V(J)$ is clear and conversely, if $p \in V(IJ)$ but $p \notin V(I)$ then there exists an $f \in I$ such that $f(p) \neq 0$. But we know that for all $g \in J$, $fg(p) = 0$ as $fg \in IJ$ hence $g(p) = 0 \in \Bbbk$ for all $g \in J$. This proves exactly $g \in V(J)$. Proof of 2 is analogously straightforward. □

Corollary 1.10. *Affine closed sets are really closed sets of a topology since, by the lemma, they are closed under finite union and arbitrary intersection. This topology is called the Zariski topology. Unless stated otherwise, we endow the closed sets of $\mathbb{A}^n$ with the subspace topology coming from the Zariski topology.*

Definition 1.11. Let Z be an arbitrary topological space. Then$Z$ is *irreducible* if in every decomposition $Z = Z_1 \cup Z_2$ using nonempty closed sets Z_1, Z_2, one of the Z_i's is the whole space Z.

Definition 1.12. An irreducible affine closed set is called an *affine variety.*

Remark 1.13. Let Z be an irreducible topological space. If $U \subseteq Z$ is a nonempty open set then it must be dense.

Example 1.14. The affine closed set $V(xy) \subseteq \mathbb{A}^2$ is not irreducible since $V(x) \cup V(y)$ is a non-trivial decomposition.

Proposition 1.15. *An affine closed set X is an affine variety if and only if $V(X)$ is a prime ideal (i.e. an ideal for which the complement is closed under multiplication).*

Proof. Direction $\Rightarrow$: Let $f, g \in \Bbbk[x_1, \dots, x_n]$ and assume that $fg \in I(X)$. Then, we necessarily have $X \subseteq V(fg) = V(f) \cup V(g)$. Therefore, X can be decomposed as

$$X = \big(X \cap V(f)\big) \cup \big(X \cap V(g)\big)$$

Hence, by irreducibility, we get that either $X = X \cap V(f)$ or $X = X \cap V(g)$ meaning that either f or g is in $I(X)$ in the corresponding cases.

Conversely, to see direction $\Leftarrow$ let $X = X_1 \cup X_2$ be a nontrivial decomposition into closed subsets. Then there exists an $f \in I(X_1)\backslash I(X)$ and $I(X_2)\backslash I(X)$ giving the product $fg \in I(X_1) \cap I(X_2)$. This implies $fg \in I(X)$ and that was the statement. $\square$

Corollary 1.16. *Every affine closed set is a finite union of affine varieties. (This is a consequence of the primary decomposition of ideals in Noetherian rings, see reference)*

Proposition 1.17. (Weak Nullstellensatz for an arbitrary field) *Let A be a finitely generated $\Bbbk$-algebra. If A is a finitely generated $\Bbbk$-algebra then it is a finite extension of $\Bbbk$.*

Corollary 1.18. *Let $\Bbbk$ be an algebraically closed field and $M \lhd \Bbbk[x_1, \dots, x_n]$ an ideal. Then $\Bbbk[x_1, \dots, x_n]/M \overset{\cong}{\to} \Bbbk$. In particular, M must have the form $(x_1 - a_1, \dots, x_n - a_n)$.*

Proof. The first part of the statement is a straightforward corollary of the previous proposition using that an algebraically closed field does not have a nontrivial finite extension, by definition. The form of M can be determined by looking at the image of each generatorx_i under the isomorphism $\Bbbk[x_1, \dots, x_n]/M \overset{\cong}{\to} \Bbbk$. Denote the images by a_i for each i, then clearly $x_i - a_i \in M$ for all $i \leq n$. However, $(x_1 - a_1, \dots, x_n - a_n)$ is already a maximal ideal (since the quotient is a field) hence we must have $(x_1 - a_1, \dots, x_n - a_n) = M$. $\square$

Corollary 1.19. *For an ideal $I \lhd \Bbbk[x_1, \dots, x_n]$, $V(I) = \emptyset$ implies $I = (1)$.*

Proof. Suppose that $I \neq (1)$ then by Zorn's lemma there exists a maximal ideal M such that $I \subseteq M$. Then, by the previous corollary, M has the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in \Bbbk$ for all i. If we translate this back to geometry, this means exactly that $(a_1, \dots, a_n) \in V(I)$ contradicting our assumption. $\square$

Proof. (of the Nullstellensatz, by the Rabinowitsch trick) First, note that the trick is not crucial in the proof, it is just to hide the process of localization which is a natural proving technique in similar questions.

Let $(f_1, \dots, f_m) = I$ by the Hilbert Basis Theorem and $g \in I(V(I))$. We need to prove that there exists an $N \in \mathbb{N}$ such that $g^N \in I(V(I))$. Consider the bigger ideal $J = (f_1, \dots, f_m, 1 - g \cdot x_{n+1}) \subseteq \Bbbk[x_1, \dots, x_n]$. One can easily check that $V(J) = \emptyset$. Indeed, if we take a point p of $V(J)$ then all the f_i's vanish on it hence $g(p) = 0$ too. But then $1 - g(p) \cdot p_{n+1} = 1 \neq 0$ so $1 - g \cdot x_{n+1}$ cannot vanish on it. This means that such a p cannot exist. Then, by Corollary 1.19, $J = (1)$ meaning that there exists $h_1, \dots, h_{m+1} \in \Bbbk[x_1, \dots, x_n, x_{n+1}]$ such that

$$\sum h_i f_i + h_{m+1}(1 - g) = 1$$

Here, we can substitute $x_{n+1} = \frac{1}{g}$ or in other words, one may consider the homomorphism $\Bbbk[x_1, \dots, x_n, x_{n+1}] \to \Bbbk(x_1, \dots, x_n)$ by $x_{n+1} \mapsto \frac{1}{g}$. So here, one can multiply by g^N for some huge $N \in \mathbb{N}$ so that we get an expression in $\Bbbk[x_1, \dots, x_n]$:

$$\sum_{i=1}^{m} \overline{h_i} \cdot f_i = g^N$$

where $\overline{h_i} = g^N \cdot h_i(x_1, \dots, x_n, \frac{1}{g}) \in \Bbbk[x_1, \dots, x_n]$. This was exactly what we wanted to prove. $\square$

Example 1.20.

1. $SL_n(\Bbbk) := \{A \in \Bbbk^{n\times n} \mid \det(A) = 1\} \subseteq \mathbb{A}^{n^2}$ is an affine variety.

2. $GL_n(\Bbbk) := \{A \in \Bbbk^{n\times n} \mid \det(A) \neq 0\} \subseteq \mathbb{A}^{n^2}$ is also an "affine variety" but in a more tricky way. Originally, it is an open set in the Zariski-topology which fails to be closed. So to define the "affine algebraic structure" on $GL_n(\Bbbk)$, one typically embeds $\mathbb{A}^{n^2}$ into $\mathbb{A}^{n^2+1}$ as $A \mapsto (A, 0)$. Then, in that space, (the image of) $GL_n(\Bbbk)$ is realized as $V\big(\det(A)x_{n^2+1} - 1\big)$. One can check that it is indeed irreducible.

Definition 1.21. Let $X \subseteq \mathbb{A}^n$ be an affine closed set. Then the *coordinate ring* of X is

$$\mathcal{A}_X := \Bbbk[x_1, \ldots, x_n]/I(X)$$

The elements of $\mathcal{A}_X$ are called *regular functions.*

Remark 1.22. The elements of $\mathcal{A}_X$ are indeed functions on X i.e. given an element $f \in \mathcal{A}_X$ and $p \in X$, it makes sense to speak about $f(p) \in \Bbbk$.

Definition 1.23. If X is an affine closed set, a *morphism* (or *regular map*) $X \to \mathbb{A}^n$ is an m-tuple $\varphi = (f_1, \ldots, f_m)$ such that $f_i \in \mathcal{A}_X$. If $P \in X$ then $\varphi(P) \in \mathbb{A}^m$. More generally, if $Y \subseteq \mathbb{A}^m$ is an affine closed set, a *morphism* $X \to Y$ is a morphism $\varphi : X \to \mathbb{A}^m$ such that $\varphi(X) \subseteq Y$. φ is called an *isomorphism* if it has an inverse function which is also a morphism.

Remark 1.24. If $\varphi : X \to Y$ is a morphism of affine closed sets, it induces a $\Bbbk$-algebra homomorphism $\varphi^* : \mathcal{A}_Y \to \mathcal{A}_X$ given by $f \mapsto f \circ \varphi$. Conversely, if $\rho : \mathcal{A}_Y \to \mathcal{A}_X$ is a $\Bbbk$-algebra homomorphism, it induces a morphism $\varphi : X \to Y$ given as follows: In $\mathcal{A}_Y$ we have the coordinate functions $\overline{x_1}, \ldots, \overline{x_m}$ of the space $Y \subseteq \mathbb{A}^m$. Hence, setting $\varphi = \big(\rho(x_1), \ldots, \rho(\overline{x_m})\big)$ gives a morphism. These two constructions are inverses to each other.

Corollary 1.25. *There is an anti-equivalence between the categories:*

$$\left\{ \begin{array}{c} \textit{affine closed sets} \\ \textit{with the mentioned morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{finitely generated } \Bbbk\textit{-algebras without nilpotent elements} \\ \textit{with } \Bbbk\textit{-algebra morphisms} \end{array} \right\}$$

which brings affine varieties into finitely generated integral domains over $\Bbbk$.

Example 1.26.

1. Consider $V(xy-1) \subseteq \mathbb{A}^2$ and the morphism $\varphi : X \to \mathbb{A}^1$ as the projection to the first coordinate. Note that then the image $\mathrm{Im}(\varphi) = \mathbb{A}^1 \backslash \{0\}$ is open and not closed. In the projective varieties setup, this will not happen, but that is a nontrivial theorem.

2. Consider the morphism $\mathbb{A}^1 \to \mathbb{A}^2$ as $t \mapsto (t^2, t^3)$. Then its image $\mathrm{Im}(\varphi) = V(y^2 - x^3) \subseteq \mathbb{A}^2$ is a closed subset. This is a bijective morphism of the varieties $\mathbb{A}^1$ and $V(y^2-x^3)$ but this is NOT an isomorphism. This fact may be hard to check geometrically, but not necessarily algebraically. Namely, isomorphic algebraic varieties must have isomorphic coordinate rings, and an isomorphism of algebraic varieties must give an isomorphism of the coordinate rings. While $\mathcal{A}_{V(y^2-x^3)} = \Bbbk[x, y]/(y^2 - x^3)$ and φ induces a non-surjective morphism
$$\Bbbk[x, y]/(y^2 - x^3) \cong \Bbbk[t^2, t^3] \hookrightarrow \Bbbk[t]$$
hence φ cannot be an isomorphism.

Proposition 1.27. *Every morphism* $\varphi : X \to Y$ *is continuous in the Zariski-topology.*

Proof. It is enough to show that for a closed subset $Z \subseteq Y$, we have $\varphi^{-1}(Z) \subseteq X$ is closed. Then one can easily check that $\varphi^{-1}(Z) = V\big(\varphi^*(J)\big)$ for the ideal J such that $Z = V(J)$. $\square$

Proposition 1.28. *Assume that $\varphi : X \to Y$ is a morphism of affine varieties. Then $\varphi^* : \mathcal{A}_Y \to \mathcal{A}_X$ is injective if and only if* $\mathrm{Im}(\varphi)$ *is dense.*

Remark 1.29. While, if $\varphi^* : \mathcal{A}_Y \to \mathcal{A}_X$ is surjective then X is a closed subvariety of Y, namely the one corresponding to the kernel of φ^*.

Proof. Direction $\Rightarrow$: If $\mathrm{Im}(\varphi)$ is not dense then there exists a proper closed subset $Z \subsetneq Y$ such that $\mathrm{Im}(\varphi) \subseteq Z$. Then, there exists a nonzero ideal $J \lhd \mathcal{A}_Y$ such that $Z = V(J)$. Now, if we take a nonzero function $f \in J$ then $\varphi^*(f) = f \circ \varphi = 0$ but that contradicts that φ^* has zero kernel. Conversely, direction $\Leftarrow$: if $f \in \mathrm{Ker}(\varphi^*)$ then $f|_{\varphi(X)} = 0$ but f is continuous and $\varphi(X) \subseteq Y$ is dense hence $f = 0$. $\square$

Suppose X is an affine variety. Then $\mathcal{A}_X$ is an integral domain. Then we can take the following:

Definition 1.30. The *function field of* X is defined as the ring of fraction $K(X) := \mathrm{Frac}(\mathcal{A}_X)$ of the coordinate ring. By definition, elements of $K(X)$ are represented by fractions of elements of $\mathcal{A}_X$ under the equivalence relation $f_1/g_1 \sim f_2/g_2$ if and only if $(f_1 g_2 - f_2 g_1)|_X = 0$.

Definition 1.31. Let $p \in X$ a point of a affine closed set. Then the *local ring* of X at p is

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in K(X) \;\middle|\; g(p) \neq 0 \right\}$$

which is clearly well-defined i.e. $f/g \in \mathcal{O}_{X,p}$ is independent of the representation of f/g. Also, notice that $\mathcal{O}_{X,p}$ has a unique maximal ideal: $M_p := \left\{ \frac{f}{g} \mid f(p) = 0 \right\}$, in particular, it is a local ring in the usual algebraic sense. If $f \in K(X)$ lies in $\mathcal{O}_{X,p}$ then we say that f is *regular* at p.

Remark 1.32. By the above, the points of X correspond bijectively to maximal ideals in $\Bbbk[x_1, \ldots, x_n]$ containing $I(X)$, which correspond to maximal ideals $\tilde{M_p}$ in $\mathcal{A}_X$. Then, in the appropriate terminology, $\mathcal{O}_{X,p} := (\mathcal{A}_X)_{\tilde{M_p}}$ i.e. the local ring at p is the *localization* of $\mathcal{A}_X$ at $\tilde{M_p}$.

Example 1.33. Let $X = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$ and $f = \frac{1-y}{x}$. Question: at which points is f regular? Obviously, where $x \neq 0$, it is. However, at $(0,1)$ we may use different representation for it as a fraction:

$$\frac{1-y}{x} = \frac{(1-y)(1+y)}{x(1+y)} = \frac{1-y^2}{x(1+y)} \overset{\text{def of } X}{=} \frac{x^2}{x(1+y)} = \frac{x}{1+y}$$

hence it is also regular at $(0,1)$. However, it is not regular at $(0,-1)$ since

$$\frac{1}{f} = \frac{x}{1-y} \in M_p \lhd \mathcal{O}_{X,p}$$

and if an inverse of a function is in the maximal ideal of the local ring then it cannot be an element of $\mathcal{O}_{X,p}$ because the invertible elements of $\mathcal{O}_{X,p}$ are exactly in the complement of M_p.

Second Lecture, 29th of September

Reminder: (From last week) We defined the notion of an $X \subseteq \mathbb{A}^n$ *affine variety*, its *coordinate ring* $\mathcal{A}_X = \Bbbk[x_1, \ldots, x_n]/I(X)$ which is an integral domain hence we could speak about the *function field* $K(X) = \mathrm{Frac}(\mathcal{A}_X)$. For each point $p \in X$ we associated a local ring $\mathcal{O}_{X,p} = (\mathcal{A}_X)_{M_p}$ i.e. it is a localization of the coordinate ring at a maximal ideal, corresponding to p. These local rings and $\mathcal{A}_X$ are naturally embedded into $K(X)$ so we can ask their relation.

Proposition 1.34. *For an affine algebraic set X we have $K(X) \supseteq \bigcap_{p\in X} \mathcal{O}_{X,p} = \mathcal{A}_X$.*

The statement can be stated as "a function is regular if and only if it is regular at every point".

Proof. Clearly, $\mathcal{O}_{X,p} \supseteq \mathcal{A}_X$ for all p by the definition of $\mathcal{O}_{X,p}$, so $\supseteq$ in the statement is true. Now, assume that $f \in K(X)$ such that $f \in \cap_{p\in X}\mathcal{O}_{X,p}$. This means that for all $p \in X$ there exist $f_p, g_p \in \mathcal{A}_X$ such that $f = \frac{f_p}{g_p}$ in $K(X)$ where $g_p(p) \neq 0$. Then let $I = \langle g_p \mid p \in X\rangle \lhd \mathcal{A}_X$. By the Noetherian condition of $\mathcal{A}_X$ it is finitely generated: $I = (g_{p_1}, \dots, g_{p_m})$ for some $p_1, \dots, p_m \in X$. By the assumption on g_p's, we get $V(I) = \emptyset$ since there is no common zero of the g_{p_i}'s. This implies $I = (1)$ by the Nullstellensatz 1.5. Writing this out gives that there exist $u_i \in \mathcal{A}_X$ $(1 \leq i \leq m)$ such that $\sum_i u_i g_{p_i} = 1$. Hence, after multiplying with f we get

$$f = \sum_i u_i g_i f = \sum_i u_i g_i \frac{f_i}{g_i} = \sum u_i f_{p_i} \in \mathcal{A}_X$$

so $f \in \mathcal{A}_X$ proving the claim. □

2 Projective varieties

Definition 2.1. Let $\Bbbk$ be an algebraically closed field (hence infinite). Then the *projective n-space* is defined as

$$\mathbb{P}^n_{\Bbbk} = \mathbb{P}^n := \big\{(a_0, \dots, a_n) \in \Bbbk^{n+1}\backslash\{0\}\big\}/\sim$$

where the equivalence relation $\sim$ is defined as follows: $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if and only if there exists a $\lambda \in \Bbbk\backslash\{0\}$ such that $a_i = \lambda b_i$ for all i.

Remark 2.2. The points of $\mathbb{P}^n_{\Bbbk}$ is in bijective correspondence with the one dimensional subspaces of $\Bbbk^{n+1}$.

Definition 2.3. We say that $f \in \Bbbk[x_0, \dots, x_n]$ *vanishes* at $p \in \mathbb{P}^n$ if $f(a_0, \dots, a_n) = 0$ for all representatives $(a_0, \dots, a_n)$ of p.

Remark 2.4. If f is homogeneous then it is equivalent to vanishing on one representative.

Definition 2.5. A *projective closed set* in $\mathbb{P}^n$ is defined as follows: Given $f_1, \dots, f_m \in \Bbbk[x_0, \dots, x_n]$, we define the *zero set* of them as

$$V(f_1, \dots, f_n) := \big\{p \in \mathbb{P}^n \mid f_i(p) = 0 \ \forall i\big\}$$

and similarly, we define the *zero set of an ideal* as

$$V(I) := \big\{p \in \mathbb{P}^n \mid f(p) = 0 \ \forall f \in I\big\}$$

A subset of $\mathbb{P}^n$ is a projective closed set if it can be written in the form of $V(I)$ for some I. These sets indeed form the closed sets of a topology which is called (again) the Zariski topology on $\mathbb{P}^n$.

Definition 2.6. A *projective variety* is an irreducible (as a topological space) projective closed set.

Definition 2.7. The *vanishing ideal* of a projective closed set is defined as

$$I(X) = \big\{f \in \Bbbk[x_0, \dots, x_n] \mid f(p) = 0 \ \forall p \in X\big\} \subseteq \Bbbk[x_0, \dots, x_n]$$

Proposition 2.8. *$I(X)$ is a homogeneous ideal, i.e. if $f \in I(X)$ and $f = f_0 + \cdots + f_d$ is a decomposition of the polynomial into homogeneous parts (f_i is homogeneous of degree i) then $f_i \in I(X)$ for all i.*

Proof. Assume that $f = f_0 + \cdots + f_d$ where f_i is homogeneous of degree i and take an arbitrary $p \in X$ represented by $(a_0, \ldots, a_n)$. Then

$$0 = f(\lambda a_0, \ldots, \lambda a_n) = \sum f_i(\lambda a_0, \ldots, \lambda a_n) = \sum \lambda^i f(\lambda a_0, \ldots, \lambda a_n)$$

for all $\lambda \in \Bbbk \setminus \{0\}$. Since $\Bbbk$ is infinite, it can only hold if $f_i(a_0, \ldots, a_d) = 0$ for all i since a one variable (λ) polynomial can have infinitely many roots only if it is zero. $\square$

Remark 2.9. If $I \subseteq \Bbbk[x_0, \ldots, x_n]$ is a homogeneous ideal then it is generated by finitely many homogeneous (!) polynomials (by the Noetherian property and the previous proposition).

Question: Does any analogue of Hilbert's Nullstellensatz hold?

In the affine case we first proved that $V(I) = \emptyset$ implies $I = (1)$. Even this part fails to remain true in the projective setting:

Example 2.10. Let $I_+ = (x_0, \ldots, x_n)$ then $I_+ \neq (1)$ but $V(I_+) = \emptyset$.

Lemma 2.11. *For a homogeneous ideal $I \lhd \Bbbk[x_0, \ldots, x_n]$ we have $V(I) \neq \emptyset$if and only if there exists an $N \in \mathbb{N}$ such that $I_+^N \subseteq I$.*

Proof. Direction $\Leftarrow$ is clear. For the converse, take a homogeneous generating system $I = (f_1, \ldots, f_m)$. By $V(I) = \emptyset$ we get that the elements $f_i(1, x_1, \ldots, x_n) \in \Bbbk[x_1, \ldots, x_n]$ $(1 \leq i \leq m)$ have no common zero in the affine space $\mathbb{A}^n$. Hence, we can apply the (affine) Nullstellensatz giving that there exist $g_i \in \Bbbk[x_1, \ldots, x_n]$ $(1 \leq i \leq m)$ such that

$$1 = \sum f_i(1, x_1, \ldots, x_n) g_i(x_1, \ldots, x_n) \in \Bbbk[x_1, \ldots, x_n]$$

Then, we may substitute $x_i \mapsto \frac{x_i}{x_0}$ into this expression. Multiplying the result by x_0^N for high enough $N \in \mathbb{N}$ will give that $x_0^N \in I$ since the right hand side is in I. This method can be applied for all the other variables $x_1, \ldots, x_n$ so for even higher $N' \in \mathbb{N}$ we get that $x_i^{N'} \in I$ $(1 \leq i \leq n)$. This means exactly that $I_+^{N''} \subseteq I$ for $N'' := n \cdot (N' - 1) + 1$ by the Pigeonhole Principle. $\square$

Theorem 2.12. (Projective Nullstellensatz) *The maps $X \mapsto I(X)$ and $I \mapsto V(I)$ induce a bijection between the following sets:*

$$\left\{ \textit{nonempty projective closed sets} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homogeneous ideal } I \textit{ such that} \\ I = \sqrt{I} \textit{ and } \nexists N \textit{ such that } I_+^N \subseteq I \end{array} \right\}$$

Proof. This follows from the lemma and the affine Nullstellensatz 1.5, using the fact that the zero sets of homogeneous ideals correspond to cones in $\mathbb{A}^{n+1}$. Details omitted. $\square$

Proposition 2.13. *$\mathbb{P}^n$ has an open covering by $n+1$ copies of $\mathbb{A}^n$.*

Proof. For all $0 \leq i \leq n$, consider the open subset

$$D_+(x_i) := \mathbb{P}^n \setminus V(x_i) = \{p = (a_0, \ldots, a_n) \mid a_i \neq 0\} \subseteq \mathbb{P}^n$$

Clearly, $\cup D_+(x_i) = \mathbb{P}^n$ by definition. Moreover, the points of $D_+(x_i)$ are in bijection with the points of $\mathbb{A}^n$ by

$$D_+(x_i) \ni (x_0, \ldots, x_n) \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_0}{x_i} \right) \in \mathbb{A}^n$$

$$D_+(x_i) \ni (t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n) \mapsfrom (t_1, \ldots, t_n) \in \mathbb{A}^n$$

This bijection is a homeomorphism with respect to the Zariski topology of $\mathbb{A}^n$ and the (induced) subspace topology of $D_+(x_i) \subseteq \mathbb{P}^n$ where the latter is also equipped with the Zariski topology. $\square$

Definition 2.14. If $X = V(I) \subseteq \mathbb{P}^n$ is a projective closed set then the subsets

$$X^{(i)} := X \cap D_+(x_i) \quad (0 \le i \le n)$$

correspond to closed sets in $\mathbb{A}^n$ via $D_+(x_i) \leftrightarrow \mathbb{A}^n$. Namely, these correspond to $V(I^{(i)})$ where

$$I^{(i)} := \left\{ f^{(i)}(t_1, \dots, t_n) := f(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n) \mid f \in I \right\}$$

Conversely, if $X_i \subseteq D_+(x_i) \cong \mathbb{A}^n$ is an affine variety for some $0 \le i \le n$ then its (Zariski-)closure in $\mathbb{P}^n$ is (tautologically) a projective closed set, called the *projective closure*. Algebraically, its vanishing ideal is defined as

$$I(\overline{X_i}) := \left\{ x_i^d g\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \mid g \in I(X_i),\ d = \deg(g) \right\}$$

giving the projective closure.

Remark 2.15. The two mentioned methods are called dehomogenization and homogenization of the polynomials. These are inverses of each other, in particular $\left(\overline{X_i}\right)^{(i)} = X_i$.

Example 2.16. Take $V(x_0^2 - x_1 x_2) \subseteq \mathbb{P}^2$ which is a projective conic plane curve. Then its dehomogenizations are

$$X^{(0)} := V(1 - t_1 t_2) \subseteq \mathbb{A}^2 \qquad X^{(1)} := V(t_1^2 - t_2) \subseteq \mathbb{A}^2 \qquad X^{(2)} := V(t_1^2 - t_2) \subseteq \mathbb{A}^2$$

that are a hyperbola, a parabola and another parabola, respectively.

At this point, we may realize that there is a problem if we want to continue to copy the affine theory in the projective case, namely the naive coordinate ring of $\mathbb{P}^n$ contains only the constant functions, that do not really describe the projective space. So instead, we first define the regular functions on subsets of the space, those work quite well in this case too.

Definition 2.17. Let X be a projective variety. Consider

$$\mathcal{R}_X := \left\{ \frac{f}{g} \mid f, g \in \Bbbk[x_0, \dots, x_n] \text{ homogeneous of the same degree} \right\}$$

Then it has a maximal ideal

$$\mathcal{M}_X := \left\{ \frac{f}{g} \in \mathcal{R}_X \mid f \in I(X) \right\}$$

The factor $K(X) := \mathcal{R}_X / \mathcal{M}_X$ is called the *function field* of X. The elements of it are represented by fraction $\frac{f}{g}$ with $\deg f = \deg g$ and two such fractions are identified if their difference vanishes on X.

Lemma 2.18. *For all i we have $K(X^{(i)}) \cong K(X)$ where $X^{(i)} = X \cap D_+(x_i) \subseteq X$.*

Proof. We can give the isomorphisms in both directions:

$$K(X) \ni \frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)} \mapsto \frac{f(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n)}{g(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_n)} \in K(X^{(i)})$$

$$K(X) \ni \frac{x_i^d f^{(i)}\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_0}{x_i}\right)}{x_i^e g^{(i)}\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_0}{x_i}\right)} \mapsfrom \frac{f^{(i)}(t_0, \dots, t_n)}{g^{(i)}(t_0, \dots, t_n)} \in K(X^{(i)})$$

where $d = \deg f^{(i)}$ and $e = \deg g^{(i)}$. It is straightforward to check that they are inverses of each other. ☐

Definition 2.19. For a point $p \in X$ we define its *local ring* as

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in K(X) \;\middle|\; g(p) \neq 0 \right\}$$

where by this we mean that those elements are included that have a representation where $g(p) \neq 0$ but (of course) not necessarily all representations have this property.

Corollary 2.20. *For $p \in X^{(i)}$ the two local rings are isomorphic, i.e. $\mathcal{O}_{X^{(i)},p^{(i)}} \cong \mathcal{O}_{X,p}$. In particular, if $p \in X^{(i)} \cap X^{(j)}$ then $\mathcal{O}_{X^{(i)},p} \cong \mathcal{O}_{X^{(j)},p}$ for all $0 \leq i, j \leq n$.*

Definition 2.21. A subset $X \subseteq \mathbb{P}^n$ that is an open subset of a projective variety is called a *quasi-projective variety.*

Remark 2.22. This is a common generalization of the affine and projective varieties. Indeed, each affine variety is an open subset of its projective closure.

Exercise 2.23. There are quasi-projective varieties that are neither affine and nor projective. Hint: Consider $\mathbb{A}^2 \backslash \{0\}$.

Definition 2.24. If X is a quasi-projective variety and $U \subseteq X$ is open then we set

$$\mathcal{O}(U) := \bigcap_{p \in U} \mathcal{O}_{X,p}$$

the *ring of regular functions* on U.

Definition 2.25. A *morphism* $\varphi : X \to Y$ of quasi-projective varieties is a continuous map such that for all open subset $U \subseteq Y$ and for all $f \in \mathcal{O}(U)$ we have $f \circ \varphi \in \mathcal{O}\big(\varphi^{-1}(U)\big)$, i.e. "regular functions pull back to regular functions".

Example 2.26.

1. If X and Y are affine varieties then this definition agrees with the definition of morphism in that setting. (This is a consequence of Proposition 1.34 stating that $\cap_{p \in X} \mathcal{O}_{X,p} = \mathcal{A}_X$ for an affine variety X.)

2. If $X \subseteq \mathbb{P}^n$ is a projective variety and $F_0, \ldots, F_n$ are homogeneous polynomials of the same degree such that $V(F_0, \ldots, F_m) \cap X = \emptyset$, then $\varphi := (F_0, \ldots, F_m)$ defines a morphism $X \to \mathbb{P}^n$. Proof: It is enough to check that $\varphi|_{X^{(i)}}$is a morphism for all i, since $X^{(i)}$ is a cover. Note that $\varphi|_{X^{(i)}}$ is the same map as

$$\left(\frac{F_0}{x_i^d}, \ldots, \frac{F_n}{x_i^d} \right)$$

for $d = \deg F_j$. However, $F_j / x_i^d \in \cap_{p \in X^{(i)}} \mathcal{O}_{X^{(i)},p}$ for all i, j still so it is indeed a morphism.

3. Consider the map $\mathbb{P}^n \to \mathbb{P}^{n-1}$, $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{n-1})$. It is not defined at $p := (0, 0, \ldots, 0, 1)$ so it is not a morphism, but if $X \subseteq \mathbb{P}^n$ is a projective variety such that $p \notin X$ then it gives a morphism $X \to \mathbb{P}^{n-1}$ by restriction.

2.1 Grassmannians

Let V be an n-dimensional $\mathbb{k}$-vector space. The problem is to classify d-dimensional linear subspaces in V using an affine or projective variety.

Remark 2.27. The core of the problem is that we are not only trying to capture the local nature of this space, e.g. we are not proving something like it is a manifold, that would be a lot easier. Now, the goal is to prove that it is a variety, i.e. it globally embeds into an affine or projective space (it turns out that, in fact, it embeds into a projective) where it is determined by polynomial equations.

Reminder: The exterior algebra of V is defined as

$$\Lambda(V) = \bigoplus_{d=0}^{\infty} \Lambda^d(V) := \bigoplus_{d=0}^{\infty} V^{\otimes d} / \langle x_1 \otimes \cdots \otimes x_d \mid \exists i \neq j,\ x_i = x_j \rangle$$

where the natural image of $x_1 \otimes \cdots \otimes x_d$ in $\Lambda(V)$ is denoted by $x_1 \wedge \cdots \wedge x_d$. If $e_1, \ldots, e_n$ is a basis in V then $e_{i_1} \wedge \cdots \wedge e_{i_d}$ for all $i_1 < \cdots < i_d$ form a basis of $\Lambda^d(V)$. Hence, $\dim \Lambda^d(V) = \binom{n}{d}$, in particular $\dim(\Lambda^{\dim V} V) = 1$. Indeed, a basis of $\Lambda^{\dim V} V$ is $\{e_1 \wedge \cdots \wedge e_n\}$ and if $v_i = \sum_{j=1}^{\dim V} \alpha_{ij} e_j$ then $v_1 \wedge \cdots \wedge v_n = \det(\alpha_{ij}) e_1 \wedge \cdots \wedge e_n$.

Definition 2.28. Let $\mathrm{Gr}_d(V)$ be the set of d-dimensional subspaces of V. Then the *Plücker embedding* is defined as

$$p_d : \mathrm{Gr}_d(V) \to \mathbb{P}\big(\Lambda^d(V)\big)$$
$$S \mapsto [\Lambda^d S]$$

where $\Lambda^d S \subseteq \Lambda^d V$ is a 1-dimensional subspace defined by the wedge-product of a basis of S. (Clearly, the image if independent of the choice of the basis, see the mentioned determinant formula.) These 1-dimensional subspaces are exactly the points of $\mathbb{P}\big(\Lambda^d(V)\big)$.

Lemma 2.29. *p_d is an injection.*

Proof. Suppose that $S_1 \neq S_2$ are subspaces of dimension d in V. Choose compatible bases in S_1 and S_2 denoted by $e_1, \ldots, e_d$ and $e_r, \ldots, e_{r+d-1}$ respectively, where $r < d$ is possible. (In details, choose a basis for the intersection and then extend to S_1 and then to a direct complement of S_2.) Now, suppose indirectly that $p_d(S_1) = p_d(S_2)$. Then we have

$$e_1 \wedge \cdots \wedge e_d = \lambda \cdot e_r \wedge \cdots \wedge e_{r+d-1}$$

Since the ordered wedge-products of the basis elements form a basis in $\Lambda^d(V)$, this can happen only if $r = 1$ so $S_1 = S_2$. □

Theorem 2.30. $\mathrm{Im}(p_d) \subseteq \mathbb{P}\big(\Lambda^d(V)\big)$ *is a projective variety.*

Definition 2.31. $\mathrm{Gr}_d(V)$ with the projective variety structure given by $\mathrm{Im}(p_d)$ is called the *Grassmannian.*

Explicitly, p_d associates to each subspace, given by an $n \times d$ matrix – corresponding to the coordinates of the generating vectors with respect to a fixed basis –, the $d \times d$ minors of its matrix.

Example 2.32. The simplest nontrivial case is $n = 4$ and $d = 2$. Here, let $e_0, \ldots, e_3$ be a basis of V and take $v_1 = \sum a_i e_i$ and $v_2 = \sum b_i e_i$ such that $\mathrm{Span}_{\mathbb{k}}(v_1, v_2) = S$. Then $p_2(S) \in \mathbb{P}^5$ (where $5 = \binom{4}{2} - 1$) is given by $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$ where the polynomials are $p_{ij} = a_i b_j - a_j b_i$. In this simple case, one can check that $\mathrm{Im}(p_2) = V(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}) \subseteq \mathbb{P}^5$ is indeed a projective variety.

Theorem 2.33.

1. $\mathrm{Im}(p_d)$ *is always given by equations of degree* 2.
2. $\mathrm{Gr}_d(V)$ *has an open covering by copies of* $\mathbb{A}^{d(n-d)}$.

Note that every element $w \in \Lambda^d(V)$ induces a point $\mathbb{P}\big(\Lambda^d(V)\big)$. It is in $\mathrm{Im}(p_d)$ if and only if $w = \lambda v_1 \wedge \dots v_d$ for independent vectors $v_i \in V$ and $\lambda \in \Bbbk^\times$.

Claim 2.34. $w \in \mathrm{Im}(p_d)$ if and only if $\dim V_w = d$ where $V_w = \mathrm{Ker}(V \to \Lambda^{d+1}V,\ v \mapsto v \wedge w)$. Otherwise, we have $\dim V_w < d$.

Proof. Let $v_1, \dots, v_m$ be a basis of V_w. Extend it to a basis of V by adjoining $v_{m+1}, \dots, v_n$. By $w \in \Lambda^d(V)$, w is a linear combination of some $v_{i_1} \wedge \dots \wedge v_{i_d}$'s. For a single tensor $v_{i_1} \wedge \dots \wedge v_{i_d}$, the multiplication by v_i gives zero if and only if $i = i_j$ for some index j, and, more importantly, if i is not among the i_j's then it maps an independent set $\{v_{i_1} \wedge \dots \wedge v_{i_d} \mid (i_1, \dots, i_d) \in I\}$ into the independent(!) set $\{v_{i_1} \wedge \dots \wedge v_{i_d} \wedge v_i \mid (i_1, \dots, i_d) \in I\}$. This means that the equation $w \wedge v_i = 0$ for all $i = 1, \dots, m$ (which is true by the choice of v_i's) can be valid only if v_i appears in every term in the linear combination defining w. (If it would not appear in a term then $\wedge w$ brings this term into a nonzero term which cannot be canceled by the others, by independence.)

Therefore, the basis vectors v_i for $1 \le i \le m$ must appear in every term of the linear combination of w. Hence, $d \ge \dim V_w$. Moreover, if $d = \dim V_w$ then $w = \lambda v_1 \wedge \dots \wedge v_d$ (in particular, $w \in \mathrm{Im}(p_d)$) since there is no more room for other v_i's in the terms, and conversely, if $w \in \mathrm{Im}(p_d)$ then we can express $w = \lambda v_1 \wedge \dots \wedge v_d$ hence $d = \dim V_w$. ☐

Proof. of Theorem 2.30: To prove the theorem, consider the map

$$\varphi : \Lambda^d(V) \to \mathrm{Hom}\big(V, \Lambda^{d+1}(V)\big)$$

$$w \mapsto (v \mapsto v \wedge w)$$

which brings $\mathrm{Im}(p_d)$ into those elements of $\mathrm{Hom}\big(V, \Lambda^{d+1}(V)\big)$ whose kernel is at least d-dimensional (by the claim). This is equivalent to $\dim \mathrm{Im}\varphi \le n-d$ which is equivalent to the vanishing of the $(n-d+1) \times (n-d+1)$ minors of the matrix of this map. These homogeneous polynomials define exactly $\mathrm{Im}(p_d) \subseteq \mathbb{P}\big(\Lambda^d(V)\big)$ by the claim 2.34. ☐

Corollary 2.35. *Let $p_{i_0,\dots,i_d}$ be the coordinates on $\mathbb{P}\big(\Lambda^d(V)\big)$. The ideal of* $\mathrm{Im}(p_d)$ *is the kernel of the map*

$$\Bbbk[p_{i_0,\dots,i_d}] \to \Bbbk[x_{ij}]_{1 \le i \le n;\ 1 \le j \le d}$$

where $p_{i_0,\dots,i_d}$ goes to the $d \times d$ matrix minor accordingly to the indices $i_0, \dots, i_d$. In other words, $\mathrm{Im}(p_d)$ *is the subset where all these matrix minors vanish.*

Third Lecture, 6th of October

3 Operations on varieties

Definition 3.1. For $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ we can define their *direct product:*

$$X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$$

i.e. it is the direct product of the underlying sets with the affine variety structure inherited from $\mathbb{A}^{n+m}$. This is indeed an affine closed set since if $X = V(f_1, \dots, f_r)$ and $Y = V(g_1, \dots, g_s)$ then we can take

$$X \times Y = V(f_1, \dots, f_r, g_1, \dots, g_s) \subseteq \mathbb{A}^{n+m}$$

where f_i's and g_j's are considered as polynomials of $n+m$ variables, that are constant in the last m or the first n variables, respectively.

Remark 3.2. If X and Y are T_1 topological spaces (i.e. 1-point sets are closed) and they are also irreducible ones then $X \times Y$ is irreducible too.

The case of projective varieties is more complicated. It is basically because $\mathbb{P}^n \times \mathbb{P}^m \neq \mathbb{P}^{n+m}$ as sets, meaning that $(x_1, \dots, x_n, 0, \dots, 0)$ represents and element on the right but not on the left. In fact, they will not even be isomorphic as quasi-projective varieties.

Definition 3.3. *Segre embedding:* It is a set map (that will turn out to be a morphism after defining the product structure properly):

$$S^{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$$

where $N = nm + n + m$, defined as

$$\big((x_0, \dots, x_n), (y_0, \dots, y_m)\big) \mapsto (x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m)$$

where the product are listed lexicographically on the right. This map is clearly injective.

Proposition 3.4. *$S^{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^N$ is Zariski-closed.*

Proof. Denote the coordinates on $\mathbb{P}^N$ by w_{ij} for $1 \leq i \leq n$ and $0 \leq j \leq m$. Then

$$W = V\big(w_{ij}w_{kl} - w_{kj}w_{il} \mid 0 \leq i, k \leq n,\ 0 \leq j, l \leq m\big)$$

gives a set that contains $S^{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ since substituting $w_{ij} = x_iy_j$ satisfies the relations. We claim that it is an equality, hence $\mathrm{Im}(S^{n,m})$ is closed.

Suppose that $Q = (w_{00}, \dots, w_{nm}) \in W$ and assume that $w_{00} \neq 0$ (by interchanging the variables properly, this can be reached). Now, let $p_1 = (w_{00}, \dots, w_{n0})$, $p_2 = (w_{00}, \dots, w_{0m})$. Then $S^{n,m}((p_1, p_2)) = Q$ by computations, because we have $w_{i0}w_{0l} = w_{00}w_{il}$ by $Q \in W$. □

Definition 3.5. The projective variety structure on the Cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is defined by $S^{n,m}$ i.e. it gets the projective variety structure as a closed subset $\mathrm{Im}(S^{n,m}) \subseteq \mathbb{P}^N$.

Recall that $\mathbb{P}^n = \cup_i D_+(x_i)$ and similarly, $\mathbb{P}^m = \cup_j D_+(y_j)$

Proposition 3.6. *$S^{n,m}\big(D_+(x_i) \times D_+(y_j)\big) = \mathrm{Im}(S^{n,m}) \cap D_+(w_{ij}) \subseteq \mathbb{P}^N$ and $S^{n,m}$ induces an isomorphism of affine varieties:*

$$D_+(x_i) \times D_+(y_j) \xrightarrow{\cong} \mathrm{Im}(S^{n,m}) \cap D_+(w_{ij})$$

Corollary 3.7. *$\mathbb{P}^n \times \mathbb{P}^m$ has an open covering by copies of $\mathbb{A}^{n+m}$'s.*

Proof. of the Proposition: It is enough to check it for $i = j = 0$ since all cases are the basically same. Then, as in the previous computation, we get that by setting $t_i = \frac{w_{i0}}{w_{00}}$ and $u_j = \frac{w_{0j}}{w_{00}}$ we get a bijection $D_+(x_0) \times D_+(y_0) \ni (t_i, u_j) \leftrightarrow (w_{00}, \dots, w_{nm}) \in D_+(w_{00})$. □

Definition 3.8. Assume that $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are quasi-projective varieties. Then we define *their product*

$$X \times Y := S^{n,m}(X \times Y) \subseteq \mathbb{P}^{nm+n+m}$$

as a quasi-projective variety.

Proposition 3.9. *$X \times Y$ is indeed a quasi-projective variety.*

Proof. It is enough to prove for the case when X and Y are projective varieties, hence else we prove for their projective closure. Now, we can use the previous propositions: $X = \cup_i\big(X \cap D_+(x_i)\big)$ and $Y = \cup_j\big(Y \cap D_+(y_j)\big)$ and $S^{n,m}(X \times Y) = \cup_{i,j}\big(S^{n,m}(X \times Y) \cap D_+(w_{ij})\big)$. Since $S^{n,m}|_{D_+(x_i) \times D_+(y_j)}$ is an isomorphism, it is enough to prove the following lemma:

Lemma 3.10. *For a topological space T and an open covering $T = \cup_{i \in I} U_i$ a subset $Z \subseteq T$ is closed if and only if $Z \cap U_i \subseteq U_i$ is closed for all $i \in I$.*

Proof. Direction $\Rightarrow$ is by definition of the induced topology. Conversely, if $Z \cap U_i \subseteq U_i$ is closed then there exists a Z_i closed such that $Z_i \cap U_i = Z \cap U_i$. We may assume that $Z_i = (Z \cap U_i) \cup T \backslash U_i$ since it is the finite union of closed sets. Hence, $Z = \cap_{i \in I} Z_i$ which is an intersection of closed sets hence closed itself. □

The proposition follows. □

Remark 3.11. The product of irreducibles varieties is also irreducible.

Theorem 3.12. *Let X be a projective variety, Y a quasi-projecitve variety, and $\varphi : X \to Y$ a morphism. Then $\varphi(X) \subseteq Y$ is closed.*

Remark 3.13. This is not true for non-projective varieties, e.g. consider the projection to $\mathbb{A}^1$ of the curve $V(xy-1) \subseteq \mathbb{A}^2$.

Corollary 3.14.

1. *If $Y = \mathbb{P}^1$ then $\varphi(X) = \mathbb{P}^1$ or a point since no other set is closed and irreducible there.*
2. *If $Y = \mathbb{A}^1$ then $\varphi(X)$ is a point since the image of $\varphi : X \to \mathbb{A}^1 \subseteq \mathbb{P}^1$ is closed in $\mathbb{P}^1$. In particular, if $f \in \mathcal{O}(X) = \cap_{p \in X} \mathcal{O}_{X,p}$ then f defines a morphism $X \to \mathbb{A}^1$ by $p \mapsto f(p)$. In this case, f must be constant by the theorem. (So the theorem shows some compactness-like phenomenon.)*
3. *If Y is affine and X is projective then every $\varphi : X \to Y$ morphism must be constant. Indeed, all the coordinate functions*
$$Y \hookrightarrow \mathbb{A}^n \twoheadrightarrow \mathbb{A}^1$$
on Y are constant on $\varphi(X)$, hence $\varphi(X)$ is a single point.
4. *If $X \subseteq \mathbb{P}^n$ is a projective variety and $X \neq \{\mathrm{pt}\}$ then for all $F \in \Bbbk[x_0, \dots, x_n]$ we have $X \cap V(F) \neq \emptyset$. In particular, any two projective plane curves intersect non-trivially. Indeed, assume that for all $p \in X$ we have $F(p) \neq 0$. Then pick a homogeneous $G \in \Bbbk[x_0, \dots, x_n]$ such that $\deg F = \deg G$ and such that $G(p) = 0$ but there is a $q \in X$ so that $G(q) \neq 0$. Then take the morphism $(F, G) : X \to \mathbb{A}^2 \backslash \{0\} \twoheadrightarrow \mathbb{P}^1$. By the assumption on G, it is not constant. Therefore, the image must be $\mathbb{P}^1$ because it is not a point, see the first corollary. This is a contradiction since then $(0, 1) \in \mathrm{Im}(\varphi)$.*

Proof. of Theorem 3.12: In the first step of the proof, for a quasi-projective variety Y we prove that $\Delta(Y) = \{(p, p) \in Y \times Y \mid p \in Y\}$ is closed. Then, as a second step, we will prove that for a projective variety X and a quasi-projective variety Y that the projection $p_2 : X \times Y \to Y$ sends closed sets to closed sets. Notice that these already imply the theorem. Indeed, if $\varphi : X \to Y$ is a morphism then define its graph as

$$\Gamma_\varphi = \left\{(p, \varphi(p)) \in X \times Y \mid p \in X\right\}$$

This $\Gamma_\varphi$is the preimage of $\Delta(Y)$ by $X \times Y \overset{(\varphi, \mathrm{id})}{\to} Y \times Y$. Now, we know that $\Delta(Y)$ is closed hence Γ_φ is closed too because (φ, id) is continuous. Then, we restrict the projection project $p_2 : X \times Y \to Y$ to the closed set Γ_φ. By the second step, the image is also closed, that is exactly $\mathrm{Im}(\varphi)$.

Remark 3.15. The first property, i.e. $\Delta(Y)$ being closed is called separatedness. The reason is that for topological spaces the analogues property (the diagonal being closed) is equivalent to Hausdorffness. So, now we prove that for quasi-projective varieties separatedness always hold. In the more general case of schemes, this is not the case.

Lemma 3.16. *Let X be a quasi-projective variety and $p \in X$. Then there exists an open $U \subseteq X$ containing p such that U is isomorphic to an affine variety.*

Corollary 3.17. *X has an open covering by affine varieties.*

Proof. of the lemma: Consider X as a subset $X \subseteq \mathbb{P}^n$. Then there exists an i so that $p \in X \cap D_+(x_i)$. Therefore, we may assume that X is an open subset of an affine variety, because it is enough to find the open neighborhood in $X \cap D_+(x_i)$. Let $Y \subseteq \mathbb{A}^n$ be this closed variety such that $Y \backslash X$ is closed, i.e. X is dense open in Y. By the Noetherian property, we may write $Y = V(f_1, \dots, f_m)$. Similarly, we have $Y \backslash X = V(f_1, \dots, f_m, g_1, \dots, g_r)$. Now, we must have an i such that $g_i(p) \neq 0$ as $p \in X$.

We claim that $D(g_i) := Y \backslash V(f_1, \dots, f_m, g_i)$ is isomorphic to an affine variety. Indeed, let $\mathbb{A}^{n+1} \supseteq Z := V(f_1, \dots, f_m, x_{n+1}g_i - 1)$ and define $\pi : Z \to \mathbb{A}^n$ as the projection $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$. Then $\mathrm{Im}(\pi) = D(g_i)$ clearly. In fact, π is an isomorphism: its inverse can be given by

$$(t_1, \dots, t_n) \mapsto \Big(t_1, \dots, t_n, \frac{1}{g_i(t_1, \dots, t_n)}\Big)$$

Obviously this is an inverse morphism. This proves the lemma since $D(g_i) \ni p$ is an open neighborhood. □

Back to the proof of the theorem: First step: By choosing an open covering $Y = \cup U_i$ where U_i is affine we can reduce the problem to the case of Y being affine. (Here, we implicitly use $\Delta(Y) = \cup \Delta(U_i)$ which is trivial.) Then, by embedding $Y \hookrightarrow \mathbb{A}^n$, we get $\Delta(Y) = Y \times Y \cap \Delta(\mathbb{A}^n)$. Therefore, it is enough to prove that $\Delta(\mathbb{A}^n) \subseteq \mathbb{A}^{n+n}$ is closed. But this is clear: using the coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ we get that

$$\Delta(\mathbb{A}^n) = V(x_i - y_j \mid 1 \leq i \leq n) \subseteq \mathbb{A}^{n+n}$$

so we got the first step.

The second step follows from the following theorem:

Theorem 3.18. (Main theorem of elimination theory, proved next time) *The projection $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ takes closed sets to closed sets.*

Indeed, take $X \subseteq \mathbb{P}^n$ embedded as a closed subset, and take $Y = \cup_i U_i$ a cover by affine closed subsets, i.e. $U_i \hookrightarrow \mathbb{A}^{n_i}$ as closed affine subsets. □

4 Rational maps

Definition 4.1. A *rational map* $\varphi : X \dashrightarrow Y$ between quasi-projective varieties is given by a morphism $\varphi : U \to Y$ where $U \subseteq X$ is open. Two pairs (U, φ) and (V, ψ) define the same rational map if $\varphi|_{U \cap V} = \psi|_{U \cap V}$.

Exercise 4.2. Let X be an affine variety and let $f_1, \dots, f_m \in K(X)$. Then $(f_1, \dots, f_m)$ defines a rational map $X \dashrightarrow \mathbb{A}^m$. Indeed, all $f_i \in \mathcal{O}(U_i)$ for some nonempty open $U_i \subseteq X$. For $U = \cap U_i$ we have $f_1, \dots, f_m \in \mathcal{O}(U)$ hence $(f_1, \dots, f_m) : U \to \mathbb{A}^m$ define a morphism.

Remark 4.3. If $\varphi : X \dashrightarrow Y$ is a rational map such that $\varphi(X) \subseteq Y$ is dense (which notion is well-defined, independently of the representative of φ) then it induces a morphism

$$\varphi^* : K(Y) \to K(X) \qquad f \mapsto f \circ \varphi$$

Definition 4.4. A rational map $\varphi : X \dashrightarrow Y$ is a *birational isomorphism* if $\varphi(X) \subseteq Y$ is dense and there exists a rational map $\psi : Y \dashrightarrow X$ with dense image such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$ where they are defined (which is necessarily a dense open subset).

Proposition 4.5. *The following are equivalent:*

1. *X and Y are birationally equivalent,*
2. *there exists a $U \subseteq X$, $V \subseteq Y$ nonempty open subsets such that $U \cong V$ as quasi-projective varieties,*

3. *$K(X) \cong K(Y)$ as $\Bbbk$-algebras.*

Proof. 2) $\Rightarrow$ 1) follows by the definition. Conversely, to prove 1) $\Rightarrow$ 2) if we denote the birational equivalences by $f: X \supseteq U_1 \to Y$ and $g: Y \supseteq V_1 \to X$ then set $U := U_1 \cap f^{-1}(V_1)$ and $V = V_1 \cap g^{-1}(U_1)$. These are the intersection of dense open sets, hence they are dense open themselves. Moreover, one can check pointwise that f and g are inverses of each other on U and V, by the definition of birational equivalence.

Similarly, 1) $\Rightarrow$ 3) is clear by the definition: we get inverse isomorphisms in the form of φ^* and ψ^*. Conversely, to prove 3) $\Rightarrow$ 1) we may assume that X and Y are affine. Then $Y \hookrightarrow \mathbb{A}^n$ where we can take the coordinate functions $t_1, \dots, t_n$. the isomorphism $K(Y) \cong K(X)$ gives the images: $t_i \overset{\psi}{\mapsto} f_i$. Then $(f_1, \dots, f_n)$ defines a rational map $\varphi : X \dashrightarrow Y$ with $\varphi(X) \subseteq Y$ dense, and similarly, we get its inverse $\psi : Y \dashrightarrow X$. The statement follows. □

Example 4.6.

1. The circle $V(x^2 + y^2 - 1)$ is birational to $\mathbb{A}^2$: by the stereographic projection.

2. Take a projective cubic $\mathbb{P}^2 \supseteq X = V(x_1^2 x_2 - x_0^3)$. Then we have a morphism $\mathbb{P}^1 \ni (t_0, t_1) \mapsto (t_1^2 t_0, t_1^3, t_0^3) \in X$ and we have its "inverse" $X \dashrightarrow \mathbb{P}^1$ defined as $(x_0, x_1, x_2) \mapsto (x_0, x_1)$ which is not defined at $(0, 0, 1)$. Conclusion: X is birational to $\mathbb{P}^1$ but its not isomorphic (proved later, reason: X has a singular point while $\mathbb{P}^1$ does not).

3. Let $\mathbb{P}^3 \supseteq X = V(x_0 x_1 - x_2 x_3) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then we can take the projection map $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2)$, $X \dashrightarrow \mathbb{P}^2$. In fact, it is birational, the rational inverse is given by $(t_0, t_1, t_2) \mapsto (t_0 t_2, t_1 t_2, t_2^2, t_0 t_1)$. Conclusion: $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$. But it is not an isomorphism: indeed, in $\mathbb{P}^2$ any two curves meet. However, in $\mathbb{P}^1 \times \mathbb{P}^1$ we have $\mathrm{pt} \times \mathbb{P}^1$ and $\mathrm{pt}' \times \mathbb{P}^1$ that do not intersect.

   Remark 4.7. We will prove that for smooth(!) projective curves, birationality implies isomorphism (see Corollary 7.27) This is something which is not true for higher dimensional varieties.

4. Take the projective line $\mathbb{P}^2 \supseteq V\big(x_1^2 x_2 - x_0(x_0 + x_2)(x_0 - x_2)\big)$. Then this is not birational to $\mathbb{P}^1$. One can prove this purely algebraically that $K(X) \ncong K(\mathbb{P}^1) \cong \Bbbk(t)$. However, we have a geometric proof as follows: (see Shafarevich) We use the fact, that will be proved later: if two projective curves are birationally isomorphic then they are isomorphic. Now, suppose indirectly that our X is birational to $\mathbb{P}^1$ so $K(X) \cong \Bbbk(t)$. Then every rational self-map $X \dashrightarrow X$ induces a $\Bbbk$-automorphism of $\Bbbk(t)$. These are given by the fractional linear maps. Such a map has at most two fixed points $\mathbb{P}^1$ since it gives a quadratic equation for the fixed points. Now, look at the equation on $D_+(x_2)$:

   $$x_1^2 x_2 = x_0(x_0 + x_2)(x_0 - x_2) \quad \rightsquigarrow \quad y^2 = x(x-1)(x+1)$$

   However, the rational map $y \mapsto -y$ and $x \mapsto x$ has at least 3 fixed points (in fact, it has 4). This is a contradiction, so these curves cannot be isomorphic.

Question 4.8. *In general, the following question is open: Given a homogeneous $F \in \mathbb{C}[x_0, \dots, x_{n+1}]$ of degree d. Consider, $V(F) \subseteq \mathbb{P}^{n+1}$. When is $V(F)$ birational to $\mathbb{P}^n$ (i.e. when is it a rational variety)? Usually, it is not, if d is large compared to n.*

Theorem 4.9. (János Kollár, 1995) *If $d > 2\left\lceil \frac{n+3}{3} \right\rceil$ then $V(F)$ is not rational for "very general F" (precisely, see below). In fact, it is not even birational to $\mathbb{P}^1 \times Z$.*

Theorem 4.10. (Totaro, 2015) *If $d > 2\left\lceil \frac{n+2}{3} \right\rceil$ then $V(F)$ is not stably rational, i.e. there exists no birational isomorphism $V(F) \times \mathbb{P}^N \overset{\cong}{\dashrightarrow} \mathbb{P}^M$ for very general F.*

Remark 4.11. In particular, this covers the case $d = 4$, $n = 3$ which was open until 2014, solved by Colliot-Thiléne, Pinetka, Voisin. In this case, $V(F)$ is not rational was known since 1970's using that the automorphisms of $V(F)$ is a finite group but for projective spaces it is not.

Definition 4.12. In the above, *"very general"* means the following: $V(F) \subseteq \mathbb{P}^{n+1}$ of degree d correspond to a point in $\mathbb{P}^{\binom{n+d+1}{d}-1}$ (the exponent is the number of homogeneous monomials of degree d in $n+1$ variables). So very general means that it the F's that satisfy the property consists to the complement of a countable union of closed subsets in $\mathbb{P}^{\binom{n+d+1}{d}-1}$. (i.e. general = outside of a closed set, and very general = outside of a countable union of closed sets).

Fourth Lecture, 13th of October

To complete the proof of Theorem 3.12, we only have to prove the following theorem:

Theorem 4.13. (identical to Theorem 3.18) *The projection $\mathbb{P}^n \times \mathbb{A}^m \xrightarrow{p_2} \mathbb{A}^m$ takes closed sets to closed sets.*

There are several proofs available. For ours, we first need the following lemma:

Lemma 4.14.

1. *Closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ can be given by zeros of polynomials $F_j \in \Bbbk[x_0, \dots, x_n, y_0, \dots, y_m]$ which are homogeneous in the x_i's and the y_j's.*

2. *Closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$ can be given by zeros of polynomials $F_j \in \Bbbk[x_0, \dots, x_n, y_0, \dots, y_m]$ homogeneous in the x_i's.*

Remark 4.15. Recall the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$. Even though $\mathbb{P}^n \times \mathbb{P}^m$ gets the variety structure by its image under this embedding, we can still give meaning to the "zeros of polynomials homogeneous in the x_i's and the y_j's".

Proof. One can clearly see that 2) follows from 1), by dehomogenization. So it is enough to prove 1).

For part 1) Consider a polynomial $F \in \Bbbk[w_{00}, \dots, w_{nm}]$ homogeneous. By setting $w_{ij} = x_i y_j$, we get a polynomial in the variables x_i's and y_j's that are of the same degree in x's and y's. Conversely, if we have a polynomial $\Bbbk[x_0, \dots, x_n, y_0, \dots, y_m]$ that are of the same degree in x's and y's, then it clearly gives a homogeneous polynomial in the w_{ij} since we can join an x_i and a y_j into w_{ij} and no x or y is left out of these "pairs" by the assumption on the x- and y-degrees. So we only have to eliminate the extra assumption on the degrees of polynomials.

Consider F that has degree d in the x_i's and degree e in the y_j's such that $F(x_0, \dots, x_n, y_0, \dots, y_m) = 0$ for all choice of the$(y_0, \dots, y_m)$. This is equivalent to saying that all the polynomials $y_j^{d-e} F(x_0, \dots, x_n, y_0, \dots, y_m) = 0$ vanish for all choice of $(y_0, \dots, y_m)$. Here, these new polynomials are homogeneous in x of degree d and also homogeneous in y of degree d. This proves the lemma. □

Proof. of Theorem 3.18=Theorem 4.13: Assume that $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$ such that $Z = V(F_1, \dots, F_M)$. Fix a point $Q = (q_1, \dots, q_m) \in \mathbb{A}^m$. Then we have $Q \notin p_2(Z)$ if and only if $V(F_1^Q, \dots, F_M^Q) = \emptyset$ where $F_i^Q := F_i(x_0, \dots, x_n, q_1, \dots, q_m)$. By the Projective Nullstellensatz 2.12, this means that there exists an $N \in \mathbb{N}$ such that $(x_0, \dots, x_n)^N \subseteq (F_1^Q, \dots, F_M^Q)$. Therefore, it is enough to see that for all $d > 0$, the sets

$$U_d = \left\{ Q \in \mathbb{A}^m \mid (x_0, \dots, x_m)^d \subseteq (F_1^Q, \dots, F_M^Q) \right\} \subseteq \mathbb{A}^m$$

are open, because then $p_2(Z) = \mathbb{A}^m \setminus \left(\cup_{d>0} U_d \right)$ is closed.

Set $d_i := \deg F_i^Q$ which is clearly independent of Q since it is always the x-degree of the original F_i. Let

$$V_D := \{\text{homogeneous polynomials of degree } D\} \subseteq \Bbbk[x_0, \dots, x_n]$$

as $\Bbbk$-vector spaces. Moreover, let

$$T_{d,Q} : V_{d-d_1} \oplus \cdots \oplus V_{d-d_M} \to V_d$$
$$(G_1, \ldots, G_M) \mapsto \sum_{i=1}^{n} F_i^Q G_i$$

$\Bbbk$-linear maps. Notice that $T_{d,Q}$ is surjective if and only if $(x_0, \ldots, x_n)^d \subseteq (F_1^Q, \ldots, F_M^Q)$.

Fix a $\Bbbk$-basis of the source and target of $T_{d,Q}$. Then $T_{d,Q}$ gets the form of a rectangular matrix, which has a minor of nonzero determinant if and only if $T_{d,Q}$ is surjective. The minors give polynomial equations in $Q = (q_1, \ldots, q_m)$, hence $\mathbb{A}^m \backslash U_d$ is Zariski-closed, defined by these equations. □

Remark 4.16. Note that the proof is nonconstructive since for example we take an infinite union of U_d's to prove that $p_2(Z)$ is closed.

5 Dimension

Remark 5.1. If $\mathbb{K} \mid \Bbbk$ is a field extension, $a_1, \ldots, a_n \in \mathbb{K}$ are algebraically independent over $\Bbbk$ if there exists a nonzero $f \in \Bbbk[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$.

Definition 5.2. The *transcendence degree* $\text{tr.deg}(\mathbb{K} \mid \Bbbk)$ of $\mathbb{K}$ over $\Bbbk$ is the cardinality of a maximal algebraically independent set over $\Bbbk$ in $\mathbb{K}$.

Fact 5.3. (not proved) *Any two maximal algebraically independent subsets in $\mathbb{K} \mid \Bbbk$ have the same cardinality.*

Definition 5.4. If X is an irreducible quasi-projective variety over $\Bbbk$ then

$$\dim(X) := \text{tr.deg}\big(K(X) \mid \Bbbk\big)$$

which must be a finite number since it is a finitely generated extension.

If X is a Zariski closed set in $\mathbb{A}^n$ or $\mathbb{P}^n$ then $\dim(X) := \max\{\dim(X_i) \mid X_i \text{ an irreducible component of } X\}$. (In this case we have no $K(X)$.)

Example 5.5. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

Definition 5.6. From now on, *curve* means a variety of dimension 1, a *surface* is a variety of dimension 2, and a *hypersurface* is a variety of codimension 1 in $\mathbb{P}^n$ or $\mathbb{A}^n$.

Proposition 5.7. *Let X be an irreducible variety. If $Y \subsetneq X$ is an irreducible closed subset then* $\dim Y < \dim X$.

Proof. We reduce to the case when X affine: It has an open covering by affine open subsets $(U_i)_{i \in I}$ that have the same function field hence they have the same dimension. Similarly for $Y \cap U_i$. In this reduced case, we have to prove the following:

Lemma 5.8. *Let A be a finitely generated integral domain over $\Bbbk$. For a nonzero prime ideal $P \subseteq A$ then*

$$\text{tr.deg}(\text{Frac}(A/P) \mid \Bbbk) < \text{tr.deg}(\text{Frac}(A) \mid \Bbbk)$$

Indeed, the left hand side is exactly the dimension of Y since $\mathcal{A}_Y = \mathcal{A}_X/P$ for some prime ideal P.

Proof. of Lemma 5.8: Let $\overline{t_1}, \ldots, \overline{t_d}$ be a maximal algebraically independent set in A/P. Lift these $\overline{t_i}$'s to $t_i \in A$. The elements $t_1, \ldots, t_d$ are algebraically independent since a dependence would give a dependence in the quotient. Now, pick any nonzero $t_0 \in P$. We claim that the elements $t_0, \ldots, t_d \in A$ are also algebraically independent over $\Bbbk$. Suppose that they are not, then there exists an $f \in \Bbbk[x_0, \ldots, x_d]$ such that $f(t_0,\dots,t_d)=0$. We may assume that f is irreducible since A is an integral domain. Then, if we take $f(t_0,\dots,t_n) \bmod P$, it gives $\overline{f}(0,\overline{t_1},\dots,\overline{t_d})=0$. Now, notice that $f(0,x_1,\dots,x_d)$ is a nonzero polynomial since else it would be divisible by x_0 but it is irreducible hence it would mean $f(x_0,\dots,x_d)=x_0$ which is, however, impossible because $f(t_0,\dots,t_d)=0$ and $t_0\neq 0$. Therefore, $f(0,x_1,\dots,x_d)$ is indeed a nonzero polynomial which contradicts the algebraic independence of $t_1,\dots,t_d$. □

The proposition follows. □

Corollary 5.9. *If $X \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ is a chain of nonempty irreducible closed subsets then $r \le \dim(X)$.*

Fact 5.10. *For a maximal such chain, $r=\dim X$.*

5.1 Tangent space, smoothness

Example 5.11. Let $X=V(f)\subseteq \mathbb{A}^2$ an affine plane curve and $p=(a,b)\in X$. Then, the equation of the tangent line is given by

$$(\partial_x f)(p)(x-a)+(\partial_y f)(p)(y-a)=0$$

This is indeed a line if the partial derivatives $(\partial_x f)(p)$ and $(\partial_y f)(p)$ are not both zero. In this case P is called a *smooth point* on X.

Remark 5.12. The older terminology for smooth point is non-singular point or (in even older literature) simple point.

Definition 5.13. Let $X=V(f_1,\dots,f_m)\subseteq \mathbb{A}^n$ an affine variety and $X\ni p=(a_1,\dots,a_n)$. Then the *tangent space* of X at p is defined as

$$T_p(X) := V\Big(\{\sum_{i=1}^n \partial_{x_i} f_j(p)(x_i-a_i) \mid j=1,\dots,m\}\Big)$$

If $p=(0,\dots,0)$ then this is a linear subspace in $\mathbb{A}^n$, otherwise it is a translation of a linear space, typically called affine subspace.

Exercise 5.14. For $X=V(y^2-x^3)\subseteq \mathbb{A}^2$ and $p=(a_1,a_2)$ we get

$$T_p(X)=V\big(3a_1^2(x-a_1)+2a_2(y-a_2)\big)$$

If we assume that $\mathrm{char}(\Bbbk)\neq 2,3$ then this is a line if $(a_1,a_2)\neq(0,0)$. But for $(0,0)$ it is zero, hence $T_{(0,0)}X=\mathbb{A}^2$, by our definition.

Remark 5.15. If $p=(0,\dots,0)$ then $T_p(X):=\mathrm{Ker}\big(T_p(f)\big)$ where $T_p(f):\mathbb{A}^n\to\mathbb{A}^m$ given by

$$T_p(f)\,:\,(x_1,\dots,x_n)\mapsto \Big(\sum_{i=1}^n \partial_{x_i} f_j(p)x_i\Big)_{j=1,\dots,m}$$

Then we get $\dim T_p(X)=n-\mathrm{rk}\big([T_p(f)]\big)=n-\mathrm{rk}\big([\partial_{x_i}f_j(p)]_{i\le n,\,j\le m}\big)$ where the matrix $[\partial_{x_i}f_j(p)]_{i\le n,\,j\le m}$ is called the *Jacobian*.

In fact, this holds for general p, just do a translation.

Corollary 5.16. *The set $\{p\in X \mid \dim T_p(X)\ge N\}\subseteq X$ is Zariski closed for all N.*

Proof. This is because $\dim_p(X)\ge N$ if and only if $\mathrm{rk}[\partial_{x_i}f_j(p)]_{i\le n,\,j\le m}\le n-N$ if and only if all the $(n-N+1)\times(n-N+1)$ minors have determinant zero. □

Definition 5.17. Let $\mathcal{O}_{X,p}$ be the local ring of X at p and $M_p \subseteq \mathcal{O}_{X,p}$ its maximal ideal. This M_p is an $\mathcal{O}_{X,p}$-module hence M_p/M_p^2 is an $\mathcal{O}_{X,p}/M_p \cong \Bbbk$-module. The name of M_p/M_p^2 is the *(Zariski) cotangent space.*

Proposition 5.18. (Zariski) *There exists an isomorphism $T_p(X)^* \cong M_p/M_p^2$ as $\Bbbk$-vector spaces.*

Corollary 5.19. *$T_p(X)$ as a $\Bbbk$-vector space does not depend on the embedding $X \hookrightarrow \mathbb{A}^n$. It is possible to define $T_p(X)$ for any quasi-projective variety.*

Observation: For every point p of X we have a maximal ideal $M \subseteq \mathcal{A}_X$, bijectively. For them, the natural injection $\rho : M/M^2 \cong M_p/M_p^2$ is an isomorphism. Indeed, if $m_1, \ldots, m_n \in M_p$ generate M_p mod M_p^2 then, by multiplying the generators by units of $\mathcal{O}_{X,p}$, we may assume that $m_i \in M$ for all i. Therefore, ρ is surjective. Moreover, $\dim M_p/M_p^2 \leq \dim M/M^2$. On the other hand, if $m_1, \ldots, m_n \in M$ are independent modulo M^2 then they are also independent modulo M_p^2.

Proof of Proposition 5.18. We may assume that $p = (0, \ldots, 0)$, else first apply a translation. We prove that $T_p(X)^* \cong M/M^2$. This is enough, by the previous observation. Define $\partial_p : M \to T_p(X)^*$ as follows: for an $f \in \Bbbk[x_1, \ldots, x_n]$ representing an element $\bar{f} \in M \subseteq \mathcal{A}_X$ set

$$\partial_p(\bar{f}) := \Big((x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} (\partial_{x_i} f)(p) x_i\Big)\Big|_{T_p(X)}$$

This does not depend on f since for $f \in I(X)$ we have $\partial_p(f) = 0$ (by the definition of $T_p(X)$) and the definition is linear. Also, if $\bar{f} \in M^2$ then one gets $\partial_p(\bar{f}) = 0$ by the Leibniz rule. Therefore, we get a map

$$\partial_p : M/M^2 \to T_p(X)^*$$

This is clearly onto since the linear function $(x_1, \ldots, x_n) \mapsto \sum a_i x_i$ for $a_i \in \Bbbk$ gives itself after derivation. To prove injectivity, assume that $\partial_p(\bar{f}) = 0$ on $T_p(X)$. This means that $\partial_p(f) = \sum_{i=1}^{n} \alpha_i \partial_p(f_i)$ for some $f_i \in I(X)$. Then, replacing f by $f - \sum \alpha_i f_i$ (which is allowed without loss of generality) we may assume that $\partial_p(f) = 0$ on the whole space $\mathbb{A}^n$. This means that f has no linear term i.e. $f \in (x_1, \ldots, x_n)^2$ hence its image in $\mathcal{A}_X$ is in M^2. □

Definition 5.20. A point p on a quasi projective variety X is called a *smooth point* if $\dim T_p(X) = \dim X$. Otherwise, p is a *singular point.* The variety X is *smooth* if and only if all $p \in X$ is smooth, otherwise X is *singular.* The set of singular points of X is denoted by X_{sing}.

Theorem 5.21. *If X is a quasi-projective variety, the smooth points form a nonempty open subset in X.*

Corollary 5.22. *For a singular point p, $\dim T_p(X) > \dim X$.*

Proof. Let $d = \dim X$. Then we have seen at Corollary 5.16that $\{p \in X \mid \dim T_p(X) \geq d\} \subseteq X$ is closed. However, by Theorem 5.21, it contains a dense open subset hence it must equal X. So if p is singular then $\dim T_p(X) > d$. □

Proof of Theorem 5.21. As a first step, we prove that the proposition is true if $X = V(f) \subseteq \mathbb{A}^n$ for some irreducible f such that $\partial_{x_n} f \neq 0$. Indeed, in this case, there exists a point $p \in X$ such that $\partial_{x_n} f(p) \neq 0$ because otherwise f divides a power of $\partial_{x_n} f$ by the Nullstellensatz, hence $f \mid \partial_{x_n} f$ since f is irreducible which is a contradiction by the degrees. Therefore, $\partial_{x_n} f(p) \neq 0$ and so $T_p(X)$ is defined by a simple nonzero linear equation meaning that $\dim T_p(X) = n - 1 = \dim X$. Hence, $V(\partial_{x_n} f)$ is a proper closed subset such that $\mathbb{A}^n \backslash V(\partial_{x_n} f)$ is a nonempty open (hence dense) subset consisting of smooth points. Note, however, that this may not agree with the set of all smooth points: we only verified that X_{sm} contains an open dense subset. The reason why X_{sm} is itself open too is proved in Remark 5.28.

If $X = V(f)$ for some irreducible f and $\partial_{x_n} f = 0$ then by a permutation of variables we can assume that $\partial_{x_n} f \neq 0$, else $\partial_{x_i} f = 0$ for all i. That is impossible in characteristic zero for a nonconstant polynomial and in positive characteristic, it means that $f = g^p$ for some p but f is assumed to be irreducible. The general case follows from the following observation: Every quasi-projective variety X is birationally isomorphic to some X of the form discussed in the first step. This means that X and $V(f)$ have isomorphic dense open subsets (see Proposition 4.5), where the set of smooth point is nonempty open (so dense), so it must be also a dense open subset in X. To prove this birational isomorphism, we need to prove that the field $K(X)$ is isomorphic to $\mathrm{Frac}\big(\Bbbk[t_1, \dots, t_m]/(f)\big)$ for some f. Hence, it is enough to prove there exist $t_1, \dots, t_m \in K(X)$ algebraically independent, such that $K(X) \mid \Bbbk(t_1, \dots, t_m)$ is a finite separable extension (since a finite separable field extension is cyclic i.e. $K(X) = \Bbbk(t_1, \dots, t_m)[t_{m+1}]/\big(f(t_{m+1})\big)$ where $f \in \Bbbk(t_1, \dots, t_n)[x]$ by the Primitive Element Theorem). Here, we did not prove that such a decomposition of $K(X) \mid \Bbbk$ exists, see Hartshorne, Chapter I, Proposition 4.8A and Theorem 5.3. The proof of the assumption $\partial_{x_{m+1}} f \neq 0$ can also be found there. □

Remark 5.23. The assumption $\partial_{x_n} f \neq 0$ is superfluous in characteristic zero but it is not in positive characteristic. See, for example $\sum_{i=1}^m x_i^p$.

Problem of resolution of singularities: Given a projective variety $X \hookrightarrow \mathbb{P}^n$, find a smooth projective variety $\tilde{X}$ and a morphism $\pi : \tilde{X} \to X$ that has a birational inverse defined on the whole smooth locus of X. Clearly, we cannot hope to define a birational inverse on all the singular points. Moreover, one should achieve that $\pi^{-1}(X_{sing})$ is a normal crossing divisor, i.e. $\pi^{-1}(X_{sing}) = \cup X_i$ where X_i is irreducible for all i and every intersection $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_r}$ are smooth varieties of codimension r.

Theorem 5.24. (Hironaka, 1960) *This always exists, if* $\mathrm{char}(\Bbbk) = 0$. *(For a modern treatment, see János Kollár: Lectures on resolution of singularities, Chapter 3)*

The problem is open in character p except in dimension at most 3.

Theorem 5.25. (de Jong, 1996) *There is a* $\pi : \tilde{X} \to X$ *with the above properties except that* $[K(\tilde{X}) \mid K(X)]$ *is a finite extension but not identical.*

Theorem 5.26. (Gabber, ~2010) *We may choose* $[K(\tilde{X}) \mid K(X)]$ *prime to any given integer except* $p = \mathrm{char}\Bbbk$.

Theorem 5.27. (Tenkin, 2015 August) *We may choose* $[K(\tilde{X}) \mid K(X)]$ *prime to* p.

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Remark 5.28. Completion to the proof of 5.21: we only proved that it contains a nonempty open subset but we need that it is itself open as well. However, for all N, $\{p \in X \mid \dim T_p X > N\} \subseteq X$ is closed. Hence, for $n = \dim X$ it contains a dense open subset, so it equals X. Therefore, the smooth locus

$$X_{\mathrm{sm}} = X \backslash \{p \in X \mid \dim T_p(X) \geq \dim X + 1\}$$

is open, as we stated.

Corollary 5.29. *If X is a smooth quasi-projective that has a resolution of singularities, then there is an an open embedding* $X \hookrightarrow \tilde{X}$ *where* $\tilde{X}$ *is a smooth projective variety.*

6 Bertini's theorem

Theorem 6.1. (Bertini's theorem) *Let* $X \subseteq \mathbb{P}^N$ *be a smooth projective variety. Then there exists a hyperplane* $H \subseteq \mathbb{P}^N$ *such that the intersection* $H \cap X$ *is again smooth, i.e. it is a disjoint union of smooth varieties. Moreover, the* H *with this property form a Zariski open subset in the dual projective space* $\left(\mathbb{P}^N\right)^*$. *In short, a general hyperplane section of a smooth variety is smooth.*

Remark 6.2.

- One can also show: (sometimes called Bertini's second theorem) There exists an H such that $H \cap X$ is smooth and connected. These form a Zariski open subset too.
- The theorem allows one to use proof by induction. For example: any smooth projective variety contains a smoothcurve.
- The theorem has the following variant:

  Theorem 6.3. (Altman, Kleiman, Coray, ~1979) *If H is allowed to be a hypersurface $V(F)$ then one can also find an H such that $H \cap X$ is smooth and contains a prescribed finite set $p_1, \ldots, p_r \in X$. Corollary: Through any $p_1, \ldots, p_r \in X$ we can find a smooth curve C passing through $p_1, \ldots, p_r$.*

- It can be shown: If X is defined over an infinite field $\mathbb{K} \subseteq \Bbbk$ then we can find an H hyperplane defined over the smaller field $\mathbb{K}$ with the properties in Theorem 6.1. Moreover:

  Theorem 6.4. (Poonen, Gabber, ~1999) *If $\mathbb{K}$ is finite, then it is not true for H being a hyperplane but it is true for H being a hypersurface.*

Problem 6.5. Assume that X is a smooth projective variety defined over $\mathbb{Q}$. Does X have a rational point over $\mathbb{Q}$?

The answer in general is no, but it has a point over $\mathbb{Q}_p$ for all but finitely many p.

Example 6.6. First, assume that $\dim X = 1$. Then we may assume that X is defined by equations with coefficients in $\mathbb{Z}$ by multiplying with the common denominator. Reducing modulo p we get a curve over $\mathbb{F}_p$. For all but finitely many p, this curve X_p is smooth. (It can be showed by computations involving the Jacobian). Moreover, one can show that (by Weil) $|X_p(\mathbb{F}_p) - p + 1| \leq 2g\sqrt{p}$ where g is the genus of X. Therefore, for high enough p, there exists a point of X_p over $\mathbb{F}_p$. Then, by Hensel's lemma, $X_p(\mathbb{F}_p) \neq \emptyset$ implies $X_p(\mathbb{Q}_p) \neq \emptyset$. Moreover, the general case of $\dim X > 1$ can be reduced to the case of $\dim X = 1$ by Bertini's theorem.

Now, we turn to the proof of Theorem 6.1. First we need the following:

Proposition 6.7. *Let $\varphi : X \to Y$ be a surjective morphism of quasi-projective varieties. Then*

1. *For all $p \in Y$ we have $\dim \varphi^{-1}(p) \geq \dim X - \dim Y$ where the dimension of the union of varieties (as $\varphi^{-1}(p)$) is the supremum of the dimensions of the components.*
2. *There exists a Zariski open $U \subseteq Y$ such that $\dim \varphi^{-1}(p) = \dim X - \dim Y$ for all $p \in U$.*

Corollary 6.8. (For proof, see Shafarevich) *Assume moreover that X and Y are projective varieties, the sets $\varphi^{-1}(p)$ are all irreducible of the same dimension and Y is irreducible. Then X is irreducible.*

Proof of Bertini's Theorem 6.1. Given $p \in X$ and $H \in \left(\mathbb{P}^N\right)^*$, observe that $X \cap H$ contains p as a smooth point if and only if $H \not\supseteq T_pX$. Moreover, $\dim(X \cap H) = \dim(X) - 1$ for highly nontrivial generalities about finitely generated $\Bbbk$-algebras (see Hartshorne Ch I. Theorem 1.8A). Recall that T_pX is defined as the projective closure of the tangent space defined in an affine open containing p.

Consider the set of bad points

$$S := \left\{(p, H) \in X \times \left(\mathbb{P}^N\right)^* \mid H \supseteq T_pX\right\}$$

Claim 6.9. We claim that S is Zariski closed in $X \times \left(\mathbb{P}^N\right)^*$.

Assuming this, consider the following projections

$$\begin{array}{ccc} & S & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & \left(\mathbb{P}^N\right)^* \end{array}$$

First, note that p_1 is onto as every T_pX is contained in some hyperplane. Moreover, if $p \in X$ then $p_1^{-1}(p)$ is a linear projective subspace of dimension $N-d-1$ by definition, where $d = \dim X$. Now, by Theorem 3.12 we get that $p_2(S) \subseteq \left(\mathbb{P}^N\right)^*$ is closed. In particular, $\dim p_2(S) \leq \dim S$. However, $\dim S = N-d-1+d$ because we have a projection of S onto X where $\dim X = d$ and every fiber has dimension $N-d-1$, so we can apply the Proposition 6.7. Therefore, $p_2(S) \subsetneq \left(\mathbb{P}^N\right)^*$ is closed, so one can pick a point $\left(\mathbb{P}^N\right)^* \backslash p_2(S)$. $\square$

Proof of Claim 6.9. Consider the map $X \to \mathrm{Gr}_d(N+1)$, $p \mapsto T_pX$. This is a morphism of projective varieties. The case where X is a hypersurface is one of the homework problems. In the general case is solved by the Plücker embedding of $\mathrm{Gr}_d(N+1)$ into a projective space. Then consider

$$Y := \left\{(Q,H) \in \mathrm{Gr}_d(N+1) \times \left(\mathbb{P}^N\right)^* \mid H \supseteq Q\right\}$$

It is enough to prove that Y is closed since then S is the inverse image of Y by the map $G \times \mathrm{id}_{\left(\mathbb{P}^N\right)^*}$: $X \times \left(\mathbb{P}^N\right)^* \to \mathrm{Gr}_d(N+1) \times \left(\mathbb{P}^N\right)^*$.

To prove that Y is closed, we prove that in general, in $\mathrm{Gr}_d(V) \times \mathrm{Gr}_e(V)$ where $e > d$ the subset of (Q,H)'s with the property $Q \subseteq H$ is closed. If $w_d = p_d(Q)$ and $w_e = p_e(H)$ where p_d and p_e are the appropriate Plücker embeddings, then we saw that $Q = \mathrm{Ker}(V \to \Lambda^{d+1}V)$ and similarly, for H and e. Hence, $Q \subseteq H$ if and only if $\mathrm{Ker}\big(V \to \Lambda^{d+1}V \times \Lambda^{e+1}V\big) = V_{w_d}$. One can prove that it is a polynomial condition for (Q,H). $\square$

Proof of Proposition 6.7. We will use the following Lemma, typically proved in commutative algebra courses.

Lemma 6.10. (Krull's Hauptidealsatz) *Let $X \subseteq \mathbb{A}^n$ be an affine variety of dimension d, $f \in \Bbbk[x_1, \ldots, x_n]$ such that $X \cap V(f) \neq \emptyset$. Then every irreducible component of $X \cap V(f)$ has dimension at least $d-1$.*

We may assume that Y is affine using the affine open covering of Y. Then fix an embedding $Y \hookrightarrow \mathbb{A}^m$ and choose $f_1 \in \Bbbk[x_1, \ldots, x_n]$ such that $V(f_1) \not\supseteq Y$ but $p \in V(f_1)$. By Krull's Hauptidealsatz, an irreducible component of $Y \cap V(f_1)$ has dimension at least $\dim(Y) - 1$ but it is a strictly contained closed subset of Y hence it must be equally $\dim(Y) - 1$. For a good choice of f_1, we can replace Y by an affine open subset that contains one irreducible component Z of $Y \cap V(f_1)$ but is disjoint from the other components of $V(f_1)$. Repeat the procedure with Z and continue. After s steps we get $f_1, \ldots, f_s$ such that $Y \cap V(f_1, \ldots, f_s) = \{p\}$. Then

$$\varphi^{-1}(p) = \left\{q \in X \mid \varphi^* f_1(q) = \cdots = \varphi^* f_s(q) = 0\right\}$$

Now, the first part of the proposition follows from the Hauptidealsatz. [I don't see all the steps here.]

For the second part of the proposition, assume that X and Y are affine. Then X has dense image hence the induced map $\varphi^* : \mathcal{A}_Y \to \mathcal{A}_X$ is an embedding. In particular, $\mathcal{A}_X = \varphi^*(\mathcal{A}_Y)[f_1, \ldots, f_s]$ for some $f_1, \ldots, f_s \in \mathcal{A}_X$ since $\mathcal{A}_X$ is a finitely generated algebra over $\Bbbk \subseteq \mathcal{A}_Y$. We may assume that $f_1, \ldots f_r$ are algebraically independent over $K(Y)$ and $f_{r+1}, \ldots, f_n$ are algebraic over $K(Y)(f_1, \ldots, f_n)$. Clearly, $\mathcal{A}_Y[f_1, \ldots, f_r]$ is the coordinate ring of some algebraic set Z and we get a factorization $(X \overset{\varphi}{\to} Y) = (X \to Z \to Y)$ since $\mathcal{A}_Y \subseteq \mathcal{A}_Y[f_1, \ldots, f_r] \subseteq \mathcal{A}_Y[f_1, \ldots, f_n] = \mathcal{A}_X$. In fact, $Z \cong Y \times \mathbb{A}^r$ by algebraic independence. In particular, the fiber of $Z \to Y$ are of dimension r (since they are affine spaces isomorphic to $\mathbb{A}^r$).

We claim that $X \to Z$ has fiber of dimension zero. For this, it is enough to prove for the case $\mathcal{A}_X \cong \mathcal{A}_Z[f]$ where f is algebraic over $K(Z)$, then we can iterate the argument for more generators. Here, every $p \in Z$ corresponds to maximal ideals $M \subseteq \mathcal{A}_Z$ be the Nullstellensatz. Moreover, the preimages of Z correspond to maximal ideals of $\mathcal{A}_X$ that contain $M\mathcal{A}_X$. These correspond to the maximal ideals of $\mathcal{A}_X/M\mathcal{A}_X$ over $(0) \in \mathcal{A}_Z/M \cong \Bbbk$. However, $\mathcal{A}_X/M\mathcal{A}_X \cong \Bbbk[\overline{f}]$ which is a finite product of finite field extensions of $\Bbbk$, but $\Bbbk$ is algebraically closed so it is a product of finitely many $\Bbbk$'s. □

7 Normal varieties, normalization

Goal: Every curve is birationally equivalent to a smooth curve.

Definition 7.1. Let $A \subseteq B$ integral domains. Then $b \in B$ is *integral* over A if and only if there exists a monic polynomial $f \in A[x]$ such that $f(b) = 0$.

The elements of B that are integral over A form a subring $\mathrm{cl}_B(A)$ of B called the *integral closure* of A in B. (Proof is omitted.) A is called integrally closed if its integral closure in $\mathrm{Frac}(A)$ is A.

Example 7.2. If A is a unique factorization domain then A is integrally closed.

Definition 7.3. A quasi-projective variety X is *normal* if and only if $\mathcal{O}_{X,p}$ is integrally closed for all $p \in X$.

Remark 7.4. If X is smooth then $\dim \mathcal{O}_{X,p} = \dim_{\Bbbk} M_p/M_p^2$ for all $p \in X$. Such local rings are called *regular local rings*. Theorem: A regular local ring is a unique factorization domain. Therefore, smooth implies normal.

Remark 7.5. An affine variety is normal if and only if $\mathcal{A}_X$ is integrally closed. This follows by that localization is exact hence it commutes with taking integral closures.

Proposition 7.6. *A curve is smooth if and only if it is normal.*

Proof. Although direction $\Rightarrow$ is the nontrivial generality we discussed in the Remark, in the case of curves, we have a simpler argument: Let $M_p \subseteq \mathcal{O}_{X,p}$ be the maximal ideal. Then $\dim_{\Bbbk} M_p/M_p^2 = \dim X = 1$ by smoothness. Then by Nakayama's lemma we get that M_p is principal.

Lemma 7.7. (Nakayama's lemma) *Let A be local ring with the maximal ideal M and N a finitely generated A-module. Then $MN = N$ implies $N = 0$.*

Here, we applied the lemma on $A = \mathcal{O}_{X,p}$, $M = M_p$ and $N = M_p/(t)$ where the image of t is a generator $\overline{t} \in M_p/M_p^2$. So in this case $\mathcal{O}_{X,p}$ is a local ring with principal maximal ideal (t). Such rings are called discrete valuation rings (in short, DVR's). Now, we can apply the following general theorem:

Fact 7.8. *A local domain A is a discrete valuation ring if and only if A is integrally closed and every nonzero prime ideal is maximal (hence it is the unique maximal ideal M).*

The fact implies the proposition since it states that $\mathcal{O}_{X,p}$ is integrally closed for all $p \in X$.

Conversely, if $\mathcal{O}_{X,p}$ is integrally closed for all $p \in X$ then, by the fact, we get that $\mathcal{O}_{X,p}$ is a DVR hence M_p is generated by 1 element, in particular $\dim M_p/M_p^2 = 1$ proving smoothness. □

Remark 7.9. In general, if X is normal then the singular locus has codimension at least 2.

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Reminder: X is a normal variety if $\mathcal{O}_{X,p}$ is integrally closed for all $p \in X$. For curves, we know that normality is equivalent to smoothness. In this case $\mathcal{O}_{X,p}$ is a discrete valuation domain (i.e. a local principal ideal domain which is not a field.)

Definition 7.10. For a quasi-projective variety X, a *normalization* of X is a pair (X^ν, p) such that X^ν is a normal quasi-projective variety, $p : X^\nu \to X$ is a finite morphism that is a birational isomorphism (see below).

Example 7.11. If we would not assume finiteness on p then we should call $\mathbb{A}^1\backslash\{0\} \hookrightarrow \mathbb{A}^1$ a normalization of $\mathbb{A}^1$ which seems silly.

Definition 7.12. A *finite* morphism $\varphi : Y \to X$ is a surjective morphism such that X has an open covering $X = \cup_i U_i$ such that U_i is isomorphic to an affine variety, for all i, $\varphi^{-1}(U_i) =: V_i$ is also isomorphic to an affine variety, and the corresponding $\varphi^* : \mathcal{A}_{U_i} \to \mathcal{A}_{V_i}$ makes $\mathcal{A}_{V_i}$ a finitely generated $\mathcal{A}_{U_i}$-module.

Remark 7.13. If $\varphi : Y \to X$ is a finite morphism then $|\varphi^{-1}(p)| < \infty$ for all $p \in X$. Indeed: first, assume that X and Y are affine. Then note that $p \in X$ corresponds to maximal ideals $M_p \subseteq \mathcal{A}_X$ and similarly, $q \in \varphi^{-1}(p)$ corresponds to maximal ideals $M_q \subseteq \mathcal{A}_Y$ such that $(\varphi^*)^{-1}M_q = M_p$, equivalently $M_q \cap \varphi^*(\mathcal{A}_U) = \varphi^*(M_p)$. Hence, q corresponds to the maximal ideals of $\mathcal{A}_Y/\varphi^*(M_p)$. This algebra is finite dimensional over the field $\mathcal{A}_X/M_p$ because $\mathcal{A}_Y$ is a finite $\mathcal{A}_X$-module. Then we are done since a finite dimensional algebra over a field has finitely many maximal ideals (see Wedderburn-Artin Theorem).

Remark 7.14. If there exists an open covering of Y as in the above definition then all the affine open coverings have this property.

Example 7.15. Let $X \subseteq \mathbb{P}^n$ be a projective variety, $p \notin X$ and take the projection from p: $\pi_p : X \to \pi_p(X) \subseteq \mathbb{P}^{n-1}$. We claim that it is a finite morphism. Indeed: We may assume that $p = (0, 0, \dots, 0, 1)$ and $X = V(f_1, \dots, f_m)$. By $p \notin X$, we know that there exists an i such that $f_i(p) \neq 0$. Since f_i is homogeneous, say of degree d then the coefficient of x_n^d must be nonzero. So we may assume that this coefficient is 1. Express f_i as

$$f_i = x_n^d + a_{n-1}(x_0, \dots, x_{n-1})x_n^{d-1} + \cdots + a_0(x_0, \dots, x_{n-1})$$

It means that x_n is integral over $\Bbbk[x_0, \dots, x_{n-1}]/I\big(\pi_p(X)\big)$ since $f_i \in I(X)$. Now, take the open covering $\big\{\pi_p(X) \cap D_+(x_j)\big\}$ of $\pi_p(X)$. This covering will satisfy the definition of a finite morphism: they are affine, the inverse images are affine too, and $\Bbbk[x_0, \dots, x_n]/I(X)$ is finitely generated over $\Bbbk[x_0, \dots, x_{n-1}]/I\big(\pi_p(X)\big)$ which is inherited for the affine coordinate rings.

Corollary 7.16. *Given $X \subseteq \mathbb{P}^n$ is a projective variety of dimension d. If $d < n$ then we just constructed a finite morphism $\pi_p : X \to \pi_p(X) =: X_{p-1} \subseteq \mathbb{P}^{n-1}$ by taking a $p \in \mathbb{P}^n\backslash X \neq \emptyset$. We may continue this process giving*

$$X \to X_{n-1} \to X_{n-2} \to \cdots \to X_d$$

where all the morphisms are finite hence dimension preserving by Proposition 6.7. So we could take a point $p \in \mathbb{P}^k\backslash X_k \neq \emptyset$ for every $d \leq k$ and the image were always a projective variety of dimension d. In the last step, it is still a projective variety (i.e. closed in $\mathbb{P}^d$) hence $X_d = \mathbb{P}^d$. The composition of finite morphisms is clearly finite, hence $X \to X_d \cong \mathbb{P}^d$ is finite.

Now, suppose that $X \subseteq \mathbb{A}^n \cong D_+(x_0) \subseteq \mathbb{P}^n$. Then we get $\pi_p(X) \subseteq D_+(x_0) \subseteq \mathbb{P}^{n-1}$ by applying the same procedure. (See: Shafarevich - Basic Algebraic Geometry, page 65) In the end, we get a finite morphism $X \to \mathbb{A}^d$. This is the Noether Normalization Lemma, proved for algebraically closed fields.

Goal: Normalization exists for an affine variety; if normalization exists for a quasi-projective variety then it is unique, finally for quasi-projective curves it always exists. (It is also true that it exists for any quasi-projective variety but we will not prove that.)

Proposition 7.17. *If X is an affine variety then the normalization $X^\nu \xrightarrow{p} X$ exists.*

Fact 7.18. *Let A be a finitely generated integral domain over a field $\Bbbk$. Take $K = \mathrm{Frac}(A)$. Let $L \mid K$ be a finite field extension, and take B as the integral closure of A in L. Then B is a finitely generated A-module. (See Eisenbud's book on Commutative algebra, the proof is harder in characteristic p.)*

Proof of Proposition 7.17. Let B be the integral closure of $\mathcal{A}_X$ in $K(X)$. The fact implies that B is a finite $\mathcal{A}_X$-module, hence it is also a finitely generated $\Bbbk$-algebra, and $\mathcal{A}_X \subseteq B$. It means that there exists an affine variety Y such that $\mathcal{A}_Y \cong B$ and the inclusion gives $p^* : \mathcal{A}_X \hookrightarrow \mathcal{A}_Y$, $p : Y \twoheadrightarrow X$. In fact, it is a finite morphism: Indeed, it can be nontrivially proved that it is surjective (see Shafarevich: Basic Algebraic Geometry, I.5.3, page 61), birational since the fraction fields are the same and we have an atlas consisting of one affine open set for both spaces where the local modules are still finite over $\mathcal{O}_{X,p}$ since they were finite before localization too. □

Proposition 7.19. *Let X be a quasi-projective variety. Assume that $p : X^\nu \to X$ exists. Then for every morphism $Y \xrightarrow{\varphi} X$ such that Y is normal and $\varphi(Y) \subseteq X$ is dense and we have a factorization*

$$\begin{array}{ccc} Y & \xrightarrow{\exists! \Psi} & X^\nu \\ & \searrow^{\varphi} & \downarrow p \\ & & X \end{array}$$

Corollary 7.20. *If (X^ν, p) exists then it is unique up to (unique) isomorphism.*

Proof of Proposition 7.19. Let $X = \cup_i U_i$ be an affine covering and $V_i = p^{-1}(U_i) \subseteq X^\nu$ also an affine covering as in the definition of finite morphism. Then $\mathcal{A}_{U_i} \hookrightarrow \mathcal{A}_{V_i}$ have the same fraction field $K(X)$. The map φ^* induces an embedding $K(X) \hookrightarrow K(Y)$. Set $W_i = \varphi^{-1}(U_i)$. We do not know yet whether they are affine. Our setup can be summarized on the following diagram

$$\begin{array}{ccccc} K(Y) & \xleftarrow[\varphi^*]{} & K(X) & \xrightarrow[\cong]{p^*} & K(X^\nu) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_{W_i,p} & \xleftarrow[\varphi^*]{} & \mathcal{A}_{U_i} & \xhookrightarrow{p^*} & \mathcal{A}_{V_i} \end{array}$$

We can consider everything By the normality of Y, $\mathcal{O}_{W_i,p}$ is integrally closed. But $\mathcal{A}_{V_i}$ is integral over $\mathcal{A}_{U_i}$ hence over $\mathcal{O}_{W_i,p}$ too (using the identifications) so $\mathcal{A}_{V_i} \subseteq \mathcal{O}_{W_i,p}$. This holds for all $p \in W_i$ hence $\mathcal{A}_{V_i} \subseteq \cap_{p \in W_i} \mathcal{O}_{W_i,p}$. If W_i would be affine, we would be done since there $\cap_{p\in W_i} \mathcal{O}_{W_i,p} = \mathcal{A}_{W_i}$ so we get a morphism $W_i \to V_i$. If W_i is not affine then we can cover it with affines: $W_i = \cup_j W_{i,j}$ where $W_{i,j}$ is affine. Moreover, one can check that as W_{ij} is dense in W_i which is dense in Y, we get that $\varphi|_{W_{ij}}$ has dense image. So, by the same argument, we get morphisms $W_{i,j} \to V_i$. The construction of $W_{ij} \to V_i$ for all i, j. was basically the restriction of $K(X^\nu) \overset{(p^*)^{-1}}{\cong} K(X) \overset{\varphi^*}{\hookrightarrow} K(Y)$ to the appropriate coordinate rings where we had to prove that the images are in $\mathcal{O}_{W_{ij}}$. Therefore, these morphisms patch together. It proves the claim. The proof of the uniqueness is omitted. □

Remark 7.21. The summary of the proof is the following: As p is a birational isomorphism, we can take the rational map $Y \xrightarrow{\varphi} X \overset{p^{-1}}{\dashrightarrow} X^\nu$. Then one has to prove that it is, in fact, a morphism, which is done by showing that $\mathcal{A}_{V_i}$ injects into $\cap_{p\in W_{ij}} \mathcal{O}_{W_i,p}$.

Theorem 7.22. *If X is a projective variety then the normalization $X^\nu \xrightarrow{p} X$ exists.*

Corollary 7.23. *The same is true for a quasi-projective variety X, just apply the theorem for the closure and check birationality and finiteness (which is a local condition) on the canonical atlas of projective spaces.*

We prove only for projective curves. Note, moreover, that the theorem solves the problem of resolution of singularities for curves since now normality implies smoothness.

Proof of Theorem 7.22. We will use the following terminology: Let (A, M_A) and (B, M_B) be local rings. Then B dominates A if $A \subseteq B$ and $M_B \cap A = M_A$.

Lemma 7.24. *If $A \subseteq B$ are discrete valuation rings,* $\mathrm{Frac}(A) = \mathrm{Frac}(B) = K$ *and B dominates A then $A = B$.*

Geometrically, if $X \subseteq \mathbb{P}^n$ is a quasi-projective curve and $\mathcal{O}_{X,p}$ dominates $\mathcal{O}_{X,q}$ for some $p, q \in X$ then $p = q$.

Proof. If there is an element $x \in M_B \backslash A$ then $x^{-1} \in M_A \subseteq M_B$ which would mean $1 = xx^{-1} \in M_B$. So $M_A = M_B$ and similarly, $B \backslash M_B = A \backslash M_A$. Hence the first claim.

For the second claim, we may assume that X is affine and $p \neq q \in X$ taking the intersection of two affine open sets around p and q. Then there exists an $f \in \mathcal{A}_X$ such that $f(q) = 0$ but $f(p) \neq 0$ since $\mathcal{A}_X$ is separating for an affine variety X. So we got a contradiction to dominating morphism. $\square$

Lemma 7.25. *Let X be a projective curve, $R \supseteq \Bbbk$ a discrete valuation domain such that* $\mathrm{Frac}(R) = K(X)$. *Then there exists a $p \in X$ such that R dominates $\mathcal{O}_{X,p}$.*

Proof. Let $x_0, \dots, x_n$ be the coordinate functions on $\mathbb{P}^n$ where an embedding $X \hookrightarrow \mathbb{P}^n$ is fixed. Then $\frac{x_i}{x_1}\Big|_X$ is an element of $K(X)$. Pick any i, j such that $v_R\left(\frac{x_i}{x_j}\right)$ is minimal where we denote the valuation of R by v_R. Notice that $v_R\left(\frac{x_i}{x_j}\right)$ must be nonpositive since otherwise we could take its additive inverse $v_R\left(\frac{x_j}{x_i}\right)$. In this case, we get $\Bbbk\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \subseteq R$ since

$$v_R\left(\frac{x_l}{x_i}\right) = v_R\left(\frac{x_l}{x_j}\right) - v_R\left(\frac{x_i}{x_j}\right) \geq 0 \quad \text{by minimality}$$

Therefore, $R \supseteq \Bbbk\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] = \mathcal{A}_{X \cap D_+(x_i)}$. (Note that here the left hand side is not a polynomial ring but the generated subring of $\mathcal{A}_{X \cap D_+(x_i)}$. Now, if we denote by M_R the maximal ideal of R then $M_R \cap \Bbbk\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$ corresponds to a prime ideal P. This is, in fact, a maximal ideal since we have the following injections

$$\Bbbk \hookrightarrow \mathcal{A}_{X \cap D_+(x_i)}/P \hookrightarrow R/M_R$$

where

$$\dim(R/M_R) \overset{5.8}{<} \dim(R) = \mathrm{trdeg}_{\Bbbk}(\mathrm{Frac}(R)) = \mathrm{trdeg}_{\Bbbk}(K(X)) = \dim(X) = 1$$

hence R/M_R is a finitely generated algebraic extension of $\Bbbk$ (hence finite) but $\Bbbk$ is algebraically closed so it is $\Bbbk$. Therefore, P is indeed maximal. It corresponds to a point $p \in X \cap D_+(x_i)$. Then we need that R dominates $\mathcal{O}_{X,p}$. As $A \backslash P \subseteq R \backslash M_R$ we get that every element of $A \backslash P$ is already invertible in R. Then one only has to check that R indeed dominates $A_P := \mathcal{A}_{X \cap D_+(x_i),p} = \mathcal{O}_{X,p}$ i.e. $M_R \cap A_P = PA_P$. For this, we use that $\otimes A_P$ is an exact functor (as A_P is flat), in particular it commutes with kernels. Therefore,

$$PA_P = \mathrm{Ker}(A \to A/P) \otimes A_P = \mathrm{Ker}(A_P \to A_P/PA_P) = \mathrm{Ker}(A_P \to R/M_R) = M_R \cap A_P$$

where a slight cheat is that the exactness of A_P only proves isomorphisms while here we need setwise equalities in $K(X)$. This can be bridged by computing the appearing isomorphisms explicitly. $\square$

Lemma 7.26. *If X is a normal quasi-projective curve (hence smooth) then a morphism $\varphi : U \to \mathbb{P}^m$ where $U \subseteq X$ is open extends to a morphism $X \to \mathbb{P}^m$.*

Corollary 7.27. *If two smooth projective curves are birational then they are isomorphic. (Follows from the lemma.)*

Remark 7.28. If X is a normal variety then every morphism $U \to \mathbb{P}^m$ where $U \subseteq X$ open extends to an open set whose complement has codimension at least 2.

Proof of Lemma 7.26. Take the closure of the image of $\varphi : X \overset{\varphi}{\dashrightarrow} \mathbb{P}^m$, i.e. $\overline{\varphi(X)} \subseteq \mathbb{P}^m$. Then we have the induced morphism $\varphi^* : K\big(\overline{\varphi(X)}\big) \hookrightarrow K(X)$. If $p \in X$ is a point where φ is not defined then for $R = \mathcal{O}_{X,p}$ we can find an i such that $\varphi^*\Big(\Bbbk\Big[\frac{x_0}{x_i}, \dots, \frac{x_m}{x_i}\Big]\Big) = \Bbbk\Big[\varphi^*\big(\frac{x_0}{x_i}\big), \dots, \varphi^*\big(\frac{x_m}{x_i}\big)\Big] \subseteq R$ by the same argument as in Lemma 7.25 (Here is the point where we use that it is a smooth curve i.e. the local rings are DVR). This means that the functions $\varphi^*\big(\frac{x_j}{x_i}\big)$ (informally this is the extension $\frac{x_j \circ \varphi}{x_i \circ \varphi}$ to the points where φ is defined) for all j are regular in an open neighborhood of p hence we may define a morphism $X \cap D_+(x_i) \to \mathbb{A}^m = D_+(x_i) \subseteq \mathbb{P}^m$. This new morphism agrees with φ on an open subset hence they agree so φ extends. □

Proof of Theorem 7.22 in dimension one: Take a covering $X = \cup_{i=1}^r U_i$ where U_i is affine for all r. For affine varieties we already proved that there exists a normalization $U_i^\nu \overset{p_i}{\to} U_i$ for all i. Hence, we get an open subset $U = \cap_{i=1}^r U_i$ such that U is smooth. We may assume that U is so small that each p_i is an isomorphism above U (since p_i's are birational isomorphisms hence isomorphisms on an open subset). Let Y_i be the projective closure of U_i^ν. Since p_i is an isomorphism we have a morphism

$$U \xrightarrow{p_i^{-1}} U_i^\nu \hookrightarrow Y_i$$

giving $\varphi : U \to \prod Y_i$. Take Y the closure of $\varphi(U)$ in $\prod Y_i$. By Lemma 7.26, φ extends to a morphism $\varphi_i : U_i^\nu \to Y$ where we identified U and $p_i^{-1}(U)$. Here, φ_i is injective as a composition of it via the projection $Y \to Y_i$ is also injective:

$$\begin{array}{ccccc} U_i^\nu & \longrightarrow & Y_i & \longleftarrow & \prod_j Y_j \\ \uparrow & & \uparrow & \nearrow & \\ U & \longrightarrow & Y & & \end{array}$$

Hence, φ_i is indeed injective.

Let $X^\nu := \cup_i \varphi_i(U_i^\nu) \subseteq Y$. We claim that $X^\nu = Y$. The claim implies the theorem because $X^\nu = Y$ is projective and $\varphi_i\big(U_i^\nu\big) \subseteq Y$ is open since it is cofinite and over U_i we have a finite morphism $\varphi_i(U_i^\nu) \cong U_i^\nu \to U_i$. Proof of the claim: Assume that $q \in Y \backslash X^\nu$. Then $\overline{\mathcal{O}_{Y,q}}$ is an integral closure of $\mathcal{O}_{Y,q}$ in $K(X)$. There are finitely many maximal ideals above the maximal ideal of $\mathcal{O}_{Y,q}$. Let M be one of them and take $R = \big(\overline{\mathcal{O}_{Y,q}}\big)_M$. This is a discrete valuation ring with function field $K(X)$. By Lemma 2, R dominates a local ring of X hence there exists a $p \in X$ such that R dominates $\mathcal{O}_{X,p}$. Let $\overline{\mathcal{O}_{X,p}}$ be the integral closure of $\mathcal{O}_{X,p}$. Localizing it at $M_R \cap \overline{\mathcal{O}_{X,p}}$ we get a discrete valuation domain dominated by R. Hence, by Lemma 1, it is equal to R. By construction, R is then a local ring of U_i^ν where i is such that $p \in U_i$. So $R = \mathcal{O}_{Y,q'}$ where $q' \in U_i^\nu \subseteq Y$. By the second part of Lemma 7.24, $q = q'$ as R dominates $\mathcal{O}_{Y,q}$. This is a contradiction. □

Remark 7.29. Generalization of this construction: Let X be a quasi-projective variety and $L \mid K(X)$ a finite extension. A normalization of X in L is a pair (X_L^ν, p) where X_L^ν is normal with function field L and $p : X_L^\nu \to X$ is a finite morphism. In particular, for $L = K(X)$, we get back the original normalization (X^ν, p). Note however, that though for X normal, X^ν was identically X, now for $L \neq K(X)$ we must get a new variety for X_L^ν even when X is normal.

Seventh Lecture, 3rd of November

Assumptions: Let X be a quasi-projective variety, $L \mid K(X)$ a finite field extension.

Recall that a normalization X_L^ν of X^ν in L is a quasi-projective variety with function field L together with a finite surjective morphism $X_L^\nu \to X$. Last time we proved the special case of the following theorem for $L = K(X)$:

Theorem 7.30.

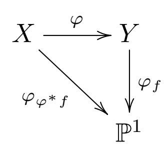
1. *If X is affine then X_L^ν exists and is affine.*
2. *Moreover, if X is a projective curve, X_L^νexists and is projective.*
3. *In general, if X_L^νexists, it is unique up to unique isomorphism.*

Proposition 7.31. *Let X be a smooth, projective curve, $f \in K(X)$ a non-constant function. Associate to f the morphism $\varphi_f : X \to \mathbb{P}^1$ (see Lemma 7.26) Then φ_f is a finite morphism.*

Proof. The map φ_f induces an embedding $K(\mathbb{P}^1) \hookrightarrow K(X)$ which is a finite extension since they are both finitely generated and have transcendence degree 1. Let Y be the normalization of $\mathbb{P}^1$ in $K(X)$. Then $\varphi_f : X \to \mathbb{P}^1$ factors through Y as $X \xrightarrow{g} Y \xrightarrow{\pi} \mathbb{P}^1$. By construction $K(X) = K(Y)$ and g is a birational morphism. Then it must be an isomorphism too since X and Y are smooth. (see Corollary 7.27) □

Corollary 7.32. *If $\varphi : X \to Y$ is a nonconstant morphism of smooth projective curves then φ is finite.*

Proof. First, φ is surjective since it is nonconstant and the image is closed and irreducible. If $f \in K(Y)$ is nonconstant then $\varphi^* f \in K(X)$ gives the following commutative diagram



where $\varphi_{\varphi^* f}$ and φ_f are finite hence φ is finite too. □

Theorem 7.33. *Let $\Bbbk$ be an algebraically closed field and $K \mid \Bbbk$ a finitely generated field extension such that $\text{tr.deg}_{\Bbbk}(K) = 1$. Then there exists a smooth projective curve X (unique up to isomorphism) such that $K(X) = K$. In this way, we obtain an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{smooth projective curves over } \Bbbk \\ \textit{with nonconstant morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{smooth projective curves over } \Bbbk \\ \textit{with finite morphisms} \end{array} \right\}$$

antiequivalent to the category

$$\left\{ \begin{array}{c} K, L \textit{finitely generated of trancendence degree 1} \\ \textit{over } \Bbbk \textit{with embeddings } K \hookrightarrow L \end{array} \right\}$$

where the first equivalence is given by the previous two propositions.

8 Divisors

Definition 8.1. Let X be a quasi-projective variety. A *Weil divisor* is a formal sum $D = \sum_{i=1}^r m_i D_i$ where $m_i \in \mathbb{Z}$ and $D_i \subseteq X$ is closed, irreducible of codimension 1. These form an abelian group $\text{Div}(X)$.

Example 8.2. On curves these are finite $\mathbb{Z}$-linear combinations of points, on surfaces the finite $\mathbb{Z}$-linear combinations of curves.

Notation: For $D \in \text{Div}(X)$ the *support* of D is defined as

$$\text{Supp}(D) := \cup_i D_i$$

where $D = \sum_{i=1}^r m_i D_i$.

Definition 8.3. A *Cartier divisor* on X is given by an open cover $\{U_i \mid i \in I\}$ of X and a system of functions $\{f_i \in K(X) \mid i \in I\}$ such that $f_i f_j^{-1}, f_j f_i^{-1} \in \mathcal{O}(U_i \cap U_j)$ for all $i \neq j$. Two systems (U_i, f_i) and (V_i, g_i) define the same Cartier divisor if $f_i g_j^{-1}$ and $f_i^{-1} g_j$ are in $\mathcal{O}(U_i \cap V_j)$ for all i, j. These form a group by the multiplication

$$\big[(U_i, f_i)_{i\in I}\big] \cdot \big[(V_j, g_j)_{j\in J}\big] = \big[(U_i \cap V_j, f_i g_j|_{U_i\cap V_j})_{(i,j)\in I\times J}\big]$$

with the unit element $\big[(X, 1)\big]$.

There exists a natural map $\operatorname{div} : K(X)^\times \to \operatorname{CaDiv}(X)$ by $f \mapsto \big[(X, f)\big]$ called the *divisor map*.

Definition 8.4. We can define the *Picard group* as $\operatorname{Pic}(X) := \operatorname{CaDiv}(X)/\operatorname{Im}(\operatorname{div})$.

Proposition 8.5. *If the local rings of X are unique factorization domains (e.g. when X is smooth). There exists a (natural) isomorphism $\operatorname{Div}(X) \overset{\cong}{\to} \operatorname{CaDiv}(X)$.*

Remark 8.6. In particular, in the above case, we may define $\operatorname{div}(f)$ as a Weil divisor $\operatorname{div}(f) = \sum m_i D_i$ for $f \in K(X)$. We say that f has a zero along D_i if $m_i > 0$ and f has a pole if $m_i < 0$.

Proof. Assume that $D = \sum m_i D_i \in \operatorname{Div}(X)$ and $p \in X$. The local ring $\mathcal{O}_{X,p}$ is a unique factorization domain, hence each D_i corresponds to (by irreducibility) a principal ideal $(t_i) \subseteq \mathcal{O}_{X,p} \subseteq K(X)$, typically called a local equation at p. Now, choose an open neighborhood U_p of p and set $f_p = \prod_i t_i^{m_i} \in K(X)^\times$. It is well-defined as $t_i \in \mathcal{O}_{X,p}$ can be represented on a small enough open neighborhood of p. Then $f_i g_j^{-1}$ and $f_i^{-1} g_j$ are regular on $U_p \cap U_{p'}$ as $(t_i) \in \mathcal{O}_{X,p}$ determines t_i up to unit multiplier.

The inverse of this construction is defined as follows: Assume now that $\big[(U_i, f_i)_{i\in I}\big] \in \operatorname{CaDiv}(X)$. We may also assume that all U_i are affine. Assume that given an irreducible subvariety $C \subsetneq X$ of codimension 1 such that $C \cap U_i \neq \emptyset$. Then we may associate the corresponding prime ideal $P_C \subseteq \mathcal{A}_{U_i}$. Then the localization $\mathcal{O}_{X,C} = \big(\mathcal{A}_{U_i}\big)_{P_C}$ is also a localization of $\mathcal{O}_{X,p}$ for all $p \in C \cap U_i$ by $M_p \supseteq P_C$. Since $\mathcal{O}_{X,p}$ is a unique factorization domain, it is, in particular, integrally closed. (In fact it is also a unique factorization domain, but that is harder to prove and we do not need it.) Therefore, $\mathcal{O}_{X,C}$ is an integrally closed too, and a Noetherian ring of dimension one (with the maximal ideal P_C), i.e. it is a discrete valuation domain.

Let $m_C = v_C(f_i)$ where v_C is the discrete valuation on $K(X)$ defined by the discrete valuation ring $\mathcal{O}_{X,C}$. It does not depend on i because if $p \in U_i \cap U_j$ then $f_i f_j^{-1}$ is a unit (hence, also in $\mathcal{O}_{X,p}$). So we get the map by associating $\sum m_C C$ to the Cartier divisor $\big[(U_i, f_i)_{i\in I}\big]$. Observe that it is a finite sum. Indeed, we may assume that U_i is a finite covering and we know that all f_i are regular and nonzero on a dense open set $V_i \subseteq X$. This means that $X \backslash V_i$ is a closed subset with finitely many components of codimension 1 (as all closed subsets have finitely many components by the Noetherian property). Therefore, m_C can be nonzero only for finitely many C. ☐

Example 8.7. Let $X = \mathbb{P}^n$ and take $C \subseteq X$ of codimension one. It corresponds to an irreducible homogeneous polynomial of degree d. Define the degree as $\widetilde{\deg}(C) := d$. This gives a homomorphism $\widetilde{\deg} : \operatorname{Div}(\mathbb{P}^n) \to \mathbb{Z}$. If $D = \sum_i m_i D_i \in \ker(\widetilde{\deg})$ then take $D_+ = \sum_{i:\, m_i>0} m_i D_i$ and $D_- = \sum_{i:\, m_i<0} (-m_i) D_i$ i.e. $D = D_+ - D_-$. Let $D_i = V(F_i)$ and define the function

$$f = \frac{\prod_{m_i>0} F_i^{m_i}}{\prod_{m_i<0} F_i^{m_i}}$$

which is a quotient of polynomials of the same degree so $f \in K(\mathbb{P}^n)$ such that $D = \operatorname{div}(f)$. It means that $\widetilde{\deg}$ gives an isomorphism $\operatorname{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. The generator of it can be chosen to a hyperplane.

Lemma 8.8. *Let X be a quasi-projective curve, $p \in X$, $f \in \mathcal{O}_{X,p} \backslash \{0\}$. Then $\mathcal{O}_{X,p}/(f)$ is a finite dimensional $\mathbb{k}$-algebra.*

Proof. It is enough to show that there exists an $n > 0$ such that $M_p^n \subseteq (f)$ as M_p^i/M_p^{i+1} are all finite dimensional algebras over $\Bbbk \cong \mathcal{O}_{X,p}/M_p$. But, if $f, g \in M_p$ then they both define the closed subset $\{p\}$ in an affine open neighborhood U_p of p since the zero locus on an affine open is a set of finitely many points and we can leave them out of U_p. Hence, by the Nullstellensatz applied on $\mathcal{O}(U_p)$ [Why is that legal? Why can we apply it on something non-closed?], there exists an n such that $g^n \in (f)$. Moreover M_p is finitely generated, hence there indeed exists a "universal" n so that $M_p^n \subseteq (f)$. [Why is it enough? M_p^n is a lot bigger than just the products of n elements.]

[Another argument: Note that $\mathcal{O}_{X,p}/(f)$ has only one prime ideal, namely $M_p/(f)$ since M_p contains only the prime ideals M_p and (0) but $f \neq 0$. Hence, the radicals of the ideals appearing in the primary decomposition of $(f)/(f) \lhd \mathcal{O}_{X,p}/(f)$ all give M_p. By $\sqrt{(0)}^N \subseteq (0)$ for a Noetherian ring, we get that $M_p/(f)$ is indeed nilpotent.] ☐

Definition 8.9. For a function $f \in K(X)^\times$ we define the *multiplicity* of f at p as $m_p(f) := \dim \mathcal{O}_{X,p}/(f_1) - \dim \mathcal{O}_{X,p}/(f_2)$ where $f = \frac{f_1}{f_2}$ for some $f_1, f_2 \in \mathcal{O}_{X,p}$. Note that it makes sense for non-smooth points too.

Example 8.10.

1. If $p \in X$ is smooth, $\mathcal{O}_{X,p}$ is a discrete valuation domain with valuation v_p. In this case, we have $m_p(f) = v_p(f)$ since we have the map $\Bbbk \cong \mathcal{O}_{X,p}/M_p \to M_p^i/M_p^{i+1}$, $a \mapsto at^i$ where $(t) = M_p$.
2. However, if p is singular, it may happen that $m_p(f) = 0$ but f is not a unit in $\mathcal{O}_{X,p}$. For example, consider $X = V(x^3 + y^3 + xy) \subseteq \mathbb{A}^2 \ni (0,0)$ and $f = \frac{x}{y}$.

Lemma 8.11.

1. *$m_p(f)$ does not depend on f_1, f_2*
2. *$m_p(f \cdot g) = m_p(f) + m_p(g)$*

Proof. Compute the dimension for the fraction $f = \frac{f_1 g}{f_2 g}$ for some $g \in \mathcal{O}_{X,p}$. Then

$$\dim_\Bbbk \mathcal{O}_{X,p}/(f_1 g) - \dim_\Bbbk \mathcal{O}_{X,p}/(f_2 g) =$$

$$= \dim_\Bbbk \mathcal{O}_{X,p}/(f_1) + \dim_\Bbbk (f_1)/(f_1 g) - \dim_\Bbbk \mathcal{O}_{X,p}/(f_2) - \dim_\Bbbk (f_2)/(f_2 g)$$

but we have an isomorphism $(f_1)/(f_1 g) \to (f_2)/(f_2 g)$ defined as $x \mapsto x\frac{f_2}{f_1}$ that have an inverse defined analogously. Hence their dimensions agree so we got $m_p(f)$. Since every two representations $\frac{f_1}{f_2} = \frac{\tilde{f}_1}{\tilde{f}_2}$ of f as a fraction have a common "refinement", the first statement follows.

The second statement is a similar computation applied on the definition. ☐

Theorem 8.12. (proof: later) *If X is a projective curve and $f \in K(X)^\times$ then $\sum_p m_p(f) = 0$.*

Construction: Let X be a quasi-projective variety such that $\mathcal{O}_{X,p}$ a unique factorization domain for all $p \in X$. Let $D \in \mathrm{Div}(X)$ and a morphism $Y \to X$ such that $\varphi(Y) \not\subseteq D$. We construct a divisor $\varphi^* D \in \mathrm{CaDiv}(Y)$ as follows:
Let $\left[(U_i, f_i)_{i\in I}\right]$ be D considered as a Cartier divisor (recall Proposition 8.5). If U_i is such that $\varphi(Y) \cap U_i \neq \emptyset$ then $\varphi(Y) \cap U_i \not\subseteq \mathrm{Supp}\big(\mathrm{div}_{U_i}(f_i)\big)$ since the latter is closed in the open subset U_i and $\varphi(Y) \not\subseteq D$. Then there exists a $p \in \varphi(Y) \cap U_i$ such that $f_i \in \mathcal{O}_{X,p}$ is a unit. Hence, $\varphi^* f = f \circ \varphi$ is regular and nonzero on an open subset of $\varphi^{-1}(U_i)$. Set $\varphi^*(D) := \left[\left(\varphi^{-1}(U_i), \varphi^* f_i\right)_{i\in I}\right]$.

Definition 8.13. If X is a curve, $D \in \mathrm{CaDiv}(X)$, $p \in X$ then $m_p(D) := m_p(f_i)$ where $p \in U_i$ and $D = \left[(U_i, f_i)_{i\in I}\right]$. This is well-defined: indeed, if $p \in U_i \cap U_j$ then $f_j f_i^{-1}$ is a unit in $\mathcal{O}_{X,p}$ hence $m_p(f_i f_j^{-1}) = 0$ i.e. $m_p(f_i) = m_p(f_j)$.

If $C_1, C_2 \subseteq \mathbb{P}^2$ are irreducible curves (not necessarily smooth) and $p \in \mathbb{P}^2$ then C_1 and C_2 correspond to ideal $I_1, I_2 \subseteq \mathcal{O}_{X,p}$. We define the *intersection multiplicity* of C_1 and C_2 at p as

$$i_p(C_1, C_2) := \dim_{\Bbbk} \mathcal{O}_{\mathbb{P}^2,p}/(I_1 + I_2)$$

Note that the definition is independent of any choice.

Lemma 8.14.

1. $i_p(C_1, C_2) = i_1(C_2, C_1)$
2. *if* $\rho : C_2 \hookrightarrow \mathbb{P}^2$ *is the inclusion then we have* $i_p(C_1, C_2) = m_p(\rho^* C_1)$ *where* C_1 *is viewed as an element of* $\mathrm{Div}(\mathbb{P}^2)$ *and* $\rho^* C_1 \in \mathrm{CaDiv}(C_2)$*.*

Note that if we would have used the second part of the lemma as a definition then symmetry would not be obvious. By the definition used above, it is.

Theorem 8.15. (Bezout's theorem) *Let* C_1*,* C_2 *be two nonidentical projective curves. Then* $\sum_{p \in C_1 \cap C_2} i_p(C_1, C_2) = \deg(C_1)\deg(C_2)$*.*

Proof. View C_1 as an element of $\mathrm{Div}(\mathbb{P}^2)$. Then the sum $\sum_{p \in C_1 \cap C_2} i_p(C_1, C_2)$ is just $\sum_{p \in C_2} m_p(\rho^* C_1)$ by part 2 of the previous lemma. However, the latter only depends on the class of C_1 in $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$. Indeed, if we take a divisor of a function and pull it back it is still a divisor of a function on C_2 hence it has multiplicity sum zero. Hence, if $\deg(C_1) = d_1$ then we may replace C_1 by $d_1 \cdot L$ where L is a line $\mathbb{P}^1 \subseteq \mathbb{P}^2$ since $\deg(L) = 1$.

Similarly, one may replace C_2 to $d_2 \cdot L'$ where $\deg(C_2) = d_2$ and L' is another line in $\mathbb{P}^2$ (the same line does not work since we cannot pull that back). Then, we only have to compute $\sum_{p \in d_1 \cdot L \cap d_2 \cdot L'} i_p(d_1 \cdot L, d_2 \cdot L') = d_1 \cdot d_2 \cdot \sum_{p \in L \cap L} i_p(L, L)$ where the intersection multiplicity of two lines is clearly 1. The claim follows. ☐

8.1 Group law on a smooth cubic

Assumptions: Let $\mathrm{char}\Bbbk \notin \{2, 3\}$ and $p, q \in \Bbbk$ such that $4p^3 + 27q^2 \neq 0$. Then

$$E = V(y^2 z = x^3 + pxz^2 + qz^3) \subseteq \mathbb{P}^2$$

is a smooth cubic. Fix a point $O \in E$. Then one can define a map $(P, Q) \mapsto P \oplus Q$ as follows: Take the line through P and Q. It intersects E in a third point R (where third can mean P or Q too, it is counted by multiplicity). Now, take the tangent line of E at O, it intersects E at a third point S. Then $P \oplus Q$ is defined as the intersection of E and the line through R and S. [missing picture]

Theorem 8.16. *The map* $(P, Q) \mapsto P \oplus Q$ *defines on* E *the structure of an abelian group with* 0 *element* O*.*

Proof. Commutativity is clear by the definition. We have the zero element O as $O + P = P$. The inverse is defined as follows: take the tangent line at O to E and let S be the third intersection. If $p \in E$ then the third intersection of E and the line through S and P will give $-P$. To prove associativity, consider $\Phi : E \to \mathrm{Pic}(E)$, $P \mapsto [P - 0]$. We claim that Φ is additive and injective. Then associativity follows since

$$\Phi\big(P \oplus (Q \oplus R)\big) = \Phi(P) + \Phi(Q) + \Phi(R) = \Phi\big((P \oplus Q) \oplus R\big)$$

where we used that associativity holds in $\mathrm{Pic}(E)$. By injectivity, we get $P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$.

To prove that Φ is indeed additive, consider the embedding $\rho : E \to \mathbb{P}^2$. Then for the line $L_1 = \overline{OR}$ and $L_2 = \overline{PQ}$ (we are using the notations of the definition of $P \oplus Q$) we have $\mathrm{div}(\rho^* L_1) = P + Q + R$ and $\mathrm{div}(\rho^* L_2) = R + O + (P \oplus Q)$ where $+$ means the addition in the Picard group. Hence,

$$\mathrm{div}\Big(\rho^* \frac{L_1}{L_2}\Big) = \mathrm{div}\Big(\rho^* L_1\Big) - \mathrm{div}\Big(\rho^* L_2\Big) = P + Q - O - (P \oplus Q)$$

where L_1/L_2 means the fraction of linear functions representing the divisors L_1 and L_2. However, $\operatorname{div}\left(\rho^* \frac{L_1}{L_2}\right) = 0$ in $\operatorname{Pic}(E)$ since it is the divisor of the fraction of the restrictions of the two linear functions. Therefore, $\Phi(P) + \Phi(Q) = \Phi(P \oplus Q)$. □

Eighth Lecture, 9th of November (Monday, "extra" lecture)

8.2 Multiplicities

Theorem 8.17. *Let X be a projective curve and $f \in K(X)^\times$. Then $\sum m_p(f) = 0$.*

Proof. First, we prove in the case when X is normal (hence smooth). Let

$$X_+ = \{p \in X \mid f \text{ is regular at } p\}$$

$$X_- = \{p \in X \mid f^{-1} \text{ is regular at } p\}$$

A decomposition of X. As we have seen last time, the nonzero rational function f gives rise to a morphism $\varphi_f : X \to \mathbb{P}^1$ (see Lemma 7.26). By Theorem 7.31 we know that φ_f is finite and surjective since it is a nonconstant morphism between smooth projective curves.

Cover $\mathbb{P}^1$ by $U_+ = \mathbb{A}^1$ and $U_- = \mathbb{P}^1 \backslash \{0\}$. Clearly, then $X_\pm = \varphi_f^{-1}(U_\pm)$ by definition. Hence, by finiteness $\mathcal{A}_{X_\pm}$ is a finitely generated $\mathcal{A}_{U_\pm}$-module. Note, moreover, that $\mathcal{A}_{U_+} \cong \mathcal{A}_{U_-} \cong \Bbbk[x]$ as $U_+ \cong U_- \cong \mathbb{A}^1$. We know that $\mathcal{A}_{X_\pm}$ is a torsion free module over $\mathcal{A}_{U_\pm}$ as the zero-set of any nonzero function in $\mathcal{A}_{U_\pm}$is closed hence its preimage in $X_\pm$ is also closed. Therefore, $\mathcal{A}_{X_\pm}$ is a finitely generated free module because $\mathcal{A}_{U_\pm}$ is a PID. We claim that

$$\sum_{p \in X_+} m_p(f) = \dim_\Bbbk \mathcal{A}_{X_+}/(f)$$

$$\sum_{p \in X_-} m_p(f) = \dim_\Bbbk \mathcal{A}_{X_-}/(f^{-1})$$

Indeed, f has finitely many zero on $X_\pm$, let us denote these by $p_1^\pm, \ldots, p_r^\pm$ and the corresponding maximal ideals by $M_{p_1^\pm}, \ldots, M_{p_r^\pm} \subseteq \mathcal{A}_{X_\pm}$. We have already seen (in Lemma 8.8) that for high enough n, $(f) \supseteq M_{p_i^\pm}^n$ for all $i = 1, \ldots, r$. By the Chinese Remainder Theorem, we have

$$\mathcal{A}_{X_\pm} \Big/ \bigcap_i M_{p_i^\pm}^n \cong \prod_{i=1}^r \left(\mathcal{A}_{X_+}/M_{p_i^\pm}^n\right) \cong \prod \left(\mathcal{O}_{X,p_i}/M_{p_i^\pm}^n\right)$$

where we can mod out both sides by (f) and hence we get the claim about the equalities.

The statement of the Theorem (in the special case when X is normal) follows by

$$\sum_{p \in X_+} m_p(f) = \dim_\Bbbk \left(\mathcal{A}_{X_+}/(f)\right) = \operatorname{rank}_{\mathcal{A}_{U_+}} \mathcal{A}_{X_+} = \operatorname{rank}_{\mathcal{A}_{(U_+ \cap U_-)}} \mathcal{A}_{X_+ \cap X_-}$$

and similarly,

$$\sum_{p \in X_-} m_p(f^{-1}) = \dim_\Bbbk \left(\mathcal{A}_{X_-}/(f^{-1})\right) = \operatorname{rank}_{\mathcal{A}_{(U_+ \cap U_-)}} \mathcal{A}_{X_+ \cap X_-}$$

So we get the first case of the Theorem by

$$\sum m_p(f) = \sum_{p \in X_+} m_p(f) - \sum_{p \in X_-} m_p(f^{-1}) = \operatorname{rank}_{\mathcal{A}_{(U_+ \cap U_-)}} \mathcal{A}_{X_+ \cap X_-} - \operatorname{rank}_{\mathcal{A}_{(U_+ \cap U_-)}} \mathcal{A}_{X_+ \cap X_-} = 0$$

Remark 8.18. Alternatively, we can finish this case from the claim $\sum_{p\in X_+} m_p(f) = \dim_{\Bbbk} \mathcal{A}_{X_+}/(f)$ as follows:

$$\sum_{p\in X_+} m_p(f) = \sum_{m_p(f)\geq 0} m_p(f) = \dim_{\Bbbk} \mathcal{A}_{X_+}/(f) = \operatorname{rank}_{\Bbbk[f]}\big(\mathcal{A}_{X_+}\big) = \big[K(X) : \Bbbk(f)\big]$$

And similarly for X_-. Note that this argument proves more than the previous one since it computes $\sum_{p\in X_+} m_p(f)$ explicitly, in terms of $K(X)$.

To prove the general case, where X is not necessarily normal, take the normalization $X^\nu \xrightarrow{\varphi} X$. It is enough to prove that for all $p \in X$ we have $\sum_{p_i\in\varphi^{-1}(p)} m_{p_i}(f\circ\varphi) = m_p(f)$. Indeed, then $\sum_{p\in X} m_p(f) = \sum_{q\in X^\nu} m_q(f\circ\varphi) = 0$ since we know the theorem for X^ν by the first case.

Take U containing p such that U is open affine such that$\varphi^{-1}(U) = V$ is also open affine. (It can be proved by generalities that such a U exists, as φ is a so called affine morphism.) Moreover, $\mathcal{A}_V$ is a finite $\mathcal{A}_U$-module since φ is finite. Let $A := \mathcal{O}_{X,p} = \big(\mathcal{A}_U\big)_{M_p}$ and $B := \big(\mathcal{A}_V\big)_{(\mathcal{A}_V\setminus M_p\mathcal{A}_V)}$. The points $\varphi^{-1}(p) \ni p_i$ correspond to maximal ideals M_{p_i} in B_i (Intuitively, B is the "local ring of V at the preimages of p"). By the same Chinese Remainder Theorem argument, we get that $\sum_{p_i} m_{p_i}(f\circ\varphi) = \dim_{\Bbbk} B/fB$. So to conclude the statement we need to prove that $\dim_{\Bbbk} A/(f) = \dim_{\Bbbk} B/fB$. Then we would get

$$\sum_{p_i} m_{p_i}(f\circ\varphi) = \dim_{\Bbbk} B/fB = \dim_{\Bbbk} A/(f) = m_p(f)$$

The equality $\dim_{\Bbbk} A/(f) = \dim_{\Bbbk} B/fB$ is proved using the following short exact sequences:

$$0 \longrightarrow fB/fA \longrightarrow B/fA \longrightarrow B/fB \longrightarrow 0$$

$$0 \longrightarrow A/fA \longrightarrow B/fA \longrightarrow B/A \longrightarrow 0$$

where one has to note that $fB/fA \cong B/A$ as a $\Bbbk$-vector space. Now, if we check that $\dim_{\Bbbk} B/A = \dim_{\Bbbk} fB/fA < \infty$ then every term in the two sequences above must be finite dimensional. Moreover, we can compute the dimensions as

$$\dim_{\Bbbk} A/fA = \dim_{\Bbbk} B/fA - \dim_{\Bbbk} B/A = \dim_{\Bbbk} B/fA - \dim_{\Bbbk} fB/fA = \dim_{\Bbbk} B/fB$$

The finiteness of $\dim_{\Bbbk} B/A$ follows by the following argument: It has a finite filtration by submodules whose quotients are A/I for some $0 \neq I \lhd A$ and where the quotients are torsion over A (since B/A is torsion over A). Since $\dim_{\Bbbk} A/I < \infty$ for all $I \neq 0$ – the same proof works as for $I = (f)$ – we get that $\dim_{\Bbbk} B/A$ is indeed finite. ☐

8.3 Back to elliptic curves

Let E be an elliptic curve. We showed last time that there exists an addition $\oplus$ on the points of E with origin O. We also considered the map $\Phi_O : E \to \operatorname{Pic}(E)$ by $P \mapsto [P - O]$ and we showed that Φ_0 is additive. If we prove that it is also injective then we get associativity of $\oplus$ because associativity already holds in the Picard group.

Proposition 8.19. *Φ_O is injective.*

Proof. Assume that it is not injective. Then $\operatorname{Ker}\Phi_0$ is nontrivial (since Φ_0 is additive, injectivity is equivalent to having zero kernel, even if we do not know associativity yet.). This means that there exists a $P \in E$ such that there exists $f \in K(E)^\times$ such that $\operatorname{div}(f) = P - O$, i.e. f has zeros only at P and O with multiplicity 1. By the argument used in Remark 8.18, we get $\big[K(E) : \Bbbk(f)\big] = 1$ hence E and $\mathbb{P}^1$ are birational. However, birationally isomorphic smooth curves are also isomorphic so $E \cong \mathbb{P}^1$. This is a contradiction. ☐

Remark 8.20. Let X be a smooth projective curve and take $D = \sum m_i p_i \in \mathrm{Div}(X)$. Define $\deg D := \sum m_i \in \mathbb{Z}$. This way, we get an induced map $\mathrm{Pic}(X) \to \mathbb{Z}$ by $D \mapsto \deg D$ because if $f \in K(X)^\times$then $\deg \mathrm{div}(f) = 0$ by Theorem 8.17.

Observe that for an elliptic curve $X = E$ we have $\mathrm{Im}(\Phi_0) = \{[D] \in \mathrm{Pic}(E) \mid \deg D = 0\} =: \mathrm{Pic}^0(E)$. So Φ_0 induces an isomorphism $E \to \mathrm{Pic}^\circ(E)$ as abelian groups.

Fact: (not proved, for $\Bbbk = \mathbb{C}$ it is a theorem of Abel and Jacobi) In general, it is possible to give $\mathrm{Pic}^\circ(X)$ the structure of a projective variety such that the group operations $\mathrm{Pic}^\circ(X) \times \mathrm{Pic}^\circ(X) \xrightarrow{\oplus} \mathrm{Pic}^\circ(X)$ and $\mathrm{Pic}^\circ(X) \xrightarrow{-} \mathrm{Pic}^\circ(X)$ are morphisms of varieties. This variety is called the Jacobian variety of the curve.

Remark 8.21. Suppose that E is defined over $\mathbb{Q}$. Let $E(\mathbb{Q})$ =points of E with $\mathbb{Q}$-coordinates. Then, by the construction, $E(\mathbb{Q})$ is an (abelian) subgroup of E. A Theorem of Mordell states that $E(\mathbb{Q})$ is finitely generated. In fact, it holds for any finite extension of $\mathbb{Q}$.

8.4 Riemann Roch Theorem

Let X be a quasi-projective variety such that for all $p \in X$, $\mathcal{O}_{X,p}$ is a unique factorization domain (hence $\mathrm{Div}(X) = \mathrm{CaDiv}(X)$).

Definition 8.22. A divisor $D \in \mathrm{Div}(X)$ is *effective* (or positive) if $D \geq 0$ i.e. $D = \sum_i m_i D_i$ where $m_i \geq 0$ for all i. Equivalently, $D \in \mathrm{CaDiv}(X)$ is effective if $D = [(U_i, f_i)]$ where f_i is regular for all i.

This defines a partial order on $\mathrm{Div}(X)$ as follows: $D_1 \geq D_2$ if and only if $D_1 - D_2 \geq 0$.

Definition 8.23. If $D \in \mathrm{Div}(X)$ then set $\mathcal{L}(D) := \{f \in K(X)^\times \mid \mathrm{div}(f) + D \geq 0\}$ is a $\Bbbk$-vector space. (So we impose some kind of a weak Mittag-Leffler condition on our functions.)

Theorem 8.24. *If X is a projective variety then* $\dim_\Bbbk \mathcal{L}(D) < \infty$*.*

Proof. (Only for the case when X is a smooth projective curve.) It is enough to prove for the case when $D \geq 0$ because if $D \geq D'$ then $\mathcal{L}(D) \supseteq \mathcal{L}(D')$, by definition. In this case, we prove that $\dim_\Bbbk \mathcal{L}(D) \leq \deg D + 1$. The proof goes by induction on $\deg D$.

For $\deg D = 0$ we have $\mathcal{L}(D) = \mathcal{O}(X) = \Bbbk$ hence the dimension is one. The induction step follows from the following observation: if $P \in X$ then $\dim_\Bbbk \mathcal{L}(D + P) \leq \dim_\Bbbk \mathcal{L}(D) + 1$. Indeed, by $D = \sum m_i P_i$ we can always write $D = D' + P$ for some point P and divisor D'. To prove the observation, let t be a generator of the maximal ideal $M_p \subseteq \mathcal{O}_{X,p}$. Let $m = v_p(D)$, and consider the $\Bbbk$-linear map

$$\mathcal{L}(D + P) \to \Bbbk \qquad f \mapsto (t^{m+1} f)(P)$$

This map has kernel exactly $\mathcal{L}(D)$ since if $t^{m+1} f$ has a zero at P then f had a pole of order at most m at P so $f \in \mathcal{L}(D)$. Consequently, $\mathcal{L}(D + P)/\mathcal{L}(D) \hookrightarrow \Bbbk$. The observation (hence the Theorem) follows. ☐

Theorem 8.25. (Weak form of the Riemann Roch Theorem) *Let X be a smooth projective curve, there exists a $g \in \mathbb{Z}_{\geq 0}$ such that for all $D \in \mathrm{Div}(X)$ with $\deg D > 2g - 2$ we have* $\dim_\Bbbk \mathcal{L}(D) = \deg D - g + 1$.

Definition 8.26. The g appearing in the theorem is called the *genus* of X.

Interpretation: (a geometric one) Let X be again arbitrary and assume that $\mathrm{Div}(X) = \mathrm{CaDiv}(X)$. For a divisor $D \in \mathrm{Div}(X)$ we define the *complete linear system* of D as

$$|D| := \{D' \geq 0 \mid [D] = [D'] \in \mathrm{Pic}(X)\}$$

Then $|D|$ is in bijection with $\mathbb{P}(\mathcal{L}(D))$. Indeed, map $|D| \ni D' \mapsto f$ where $D' = \mathrm{div}(f) + D$ for some $f \in K(X)^\times$ (it exists by $[D] = [D']$). Note that $\mathrm{div}(f)$ determines f up to constant multiplier. (Indeed, their quotient is a regular function on the whole space hence constant.) The reverse map is even more simple.

Definition 8.27. A *linear system* on X is a projective linear subspace $L \subseteq |D|$ for some D. Moreover, L is *base point free* if $\cap_{D' \in L} \mathrm{Supp}(D') = \emptyset$.

Example 8.28.

1. Let $X = \mathbb{P}^n$ and $H \subseteq \mathbb{P}^n$ a hyperplane. Then $|H|$ is the set of hyperplanes in $\mathbb{P}^n$. It is base point free.
2. For general X, given a morphism $\varphi : X \to \mathbb{P}^n$ such that the image is not in a hyperplane, we can pull back the previous linear system giving $\varphi^*|H| = \{\varphi^* D \mid |D \in |H|\}$ which is again a base point free system on X. Clearly, it is again a linear system since pullback of an effective divisor (i.e. one given by regular functions) is again effective because the pullback of regular functions is trivially regular. Similarly, one can verify the base point free property.

Proposition 8.29. *The construction of example 2) induces a bijection*

$$\{\textit{morphisms } X \to \mathbb{P}^n \textit{ with image not in a hyperplane}\} \longleftrightarrow \{\textit{base point free linear systems on } X\}$$

Proof. Consider a linear system $L \subseteq \mathbb{P}(\mathcal{L}(D))$. Let $f_0, \ldots, f_n \in \mathcal{L}(D)$ be a basis of L. Let $U_i := X \backslash \mathrm{Supp}(\mathrm{div}(f) + D) \subseteq X$. These are open by the definition and $X = \bigcup_i U_i$ as L is base point free. Consider the maps

$$U_i \to D_+(x_i) \subseteq \mathbb{P}^n \qquad P \mapsto \left(\frac{f_0}{f_i}(P), \ldots, \frac{f_{i-1}}{f_i}(P), \frac{f_{i+1}}{f_i}(P), \ldots, \frac{f_n}{f_i}(P)\right)$$

Observe that we have $\frac{f_j}{f_i} \in \mathcal{O}(U_i)$ since

$$\mathrm{div}\left(\frac{f_j}{f_i}\right) = \mathrm{div}(f_j) + D - \big(\mathrm{div}(f_i) + D\big)$$

where $\mathrm{div}(f_i) + D = 0$ since we are on $U_i = X \backslash \mathrm{Supp}\big(\mathrm{div}(f) + D\big)$ and $\mathrm{div}(f_j) + D \geq 0$ by definition. Hence, $\mathrm{div}\big(\frac{f_j}{f_i}\big) \geq 0$ meaning that it is regular.

Moreover, the above maps coincide on $U_i \cap U_j$ so we can glue them together giving a $\varphi : X \to \mathbb{P}^n$. By construction, $\varphi^* V(x_i) = X \backslash U_i$ so $\varphi^*|H| = L$. ☐

Remark 8.30. Note that without the base point free assumption the construction still works but gives a rational map $\varphi : X \supseteq \cup_i U_i \to \mathbb{P}^n$ instead.

Assumption: From now on, X is a smooth projective curve.

Theorem 8.31. *Assume $D \in \mathrm{Div}(X)$ such that $|D|$ and $|D - P|$ are base point free for all $P \in X$. Then $|D|$ corresponds to a morphism $\varphi_{|D|} : X \to \mathbb{P}^n$ mapping X isomorphically onto its image.*

Corollary 8.32. *If $\deg(D) > 2g$ then $\varphi_{|D|}$ embeds X isomorphically in $\mathbb{P}^n$.*

Proof. of Corollary 8.32: We have to check the assumptions of Theorem 8.31: $|D|$ is base point free if and only if $\mathcal{L}(D) \neq \mathcal{L}(D - P)$ for all $P \in X$. We have seen that $\dim \mathcal{L}(D) - \dim \mathcal{L}(D - P) \leq 1$ (see the proof of Theorem 8.24) so $\mathcal{L}(D) \neq \mathcal{L}(D - P)$ can happen only if it is exactly one. Similarly, $D - P$ is base point free if and only if $\dim \mathcal{L}(D-P) - \dim \mathcal{L}(D-P-Q) = 1$ for all P, Q (possibly $P = Q$). Since $\deg(D-P-Q) > 2g-2$, by Riemann Roch, we get that

$$\dim \mathcal{L}(D) = \deg D - g + 1 = \deg(D - P) - g + 2 = \dim \mathcal{L}(D - P) + 1$$

and similarly, it equals to $\dim \mathcal{L}(D - P - Q) + 2$. Hence, we can indeed apply Theorem 8.31 which proves the statement. ☐

Remark 8.33. In the above setting, let $D' \in |D|$ where $D' = \varphi_{|D|}^* H$ for some hyperplane $H \subseteq \mathbb{P}^n$ by Proposition 8.29. Then $\deg(D')$ $(= \deg(D))$ is the number of points in $\varphi_{|D|}(X) \cap H$ counted with multiplicities. This shows why the degree is a natural generalization of the naive degree notion of polynomials on surfaces.

Corollary 8.34. *If $P \in X$ and $D = (2g+1)P$ then $|D|$ is an embedding of X as a curve of degree $2g+1$ in $\mathbb{P}^{g+1}$. In particular, if $g = 0$ then $X \cong \mathbb{P}^1$, if $g = 1$ then X is isomorphic to a smooth cubic curve in $\mathbb{P}^2$.*

Remark 8.35. In fact, any projective curve can be embedded in $\mathbb{P}^3$ by a similar iterated projection method as Whitney's theorem works in Differential topology.

Proof. of Theorem 8.31: Let $|D|$ be a base point free linear system. Then we have already constructed a map $\varphi_{|D|} : X \to \mathbb{P}^n$. We claim that it is injective. Indeed, if $P \neq Q \in X$ then there exists $D' \in |D - P|$ such that $Q \notin \operatorname{Supp}(D')$ as $|D-P|$ is base point free. Clearly, $|D-P| \subsetneq |D|$ hence there exists a hyperplane $H' \subseteq \mathbb{P}^n$ such that $D' = \varphi_{|D|}^*(H')$, in particular, $\varphi_{|D|}(P) \in H'$ but $\varphi_{|D|}(Q) \notin H'$. This proves injectivity.

We still have to show that $\varphi_{|D|} : X \to \varphi_{|D|}(X)$ is an isomorphism on local rings. For this, we show that it induces isomorphisms on every local ring. By the definition of morphism of varieties, the latter is enough. For this goal, let us turn to (the restriction of) the induced map on the function fields:

$$\varphi_{|D|}^* : K\big(\varphi_{|D|}(X)\big) \supseteq \mathcal{O}_{\varphi_{|D|}(X), \varphi_{|D|}(P)} \to \mathcal{O}_{X,P} \subseteq K(X)$$

Notice that $X \to \varphi_{|D|}(X)$ is a finite morphism because there exists a factorization $X \to \big(\varphi_{|D|}(X)\big)^{\nu} \to \varphi_{|D|}(X)$ since X is normal so $X \to \varphi_{|D|}(X)$ factors through the normalization of $\varphi_{|D|}(X)$. Here, both morphisms are finite hence the composition is also finite. Therefore, $\mathcal{O}_{X,P}$ is a finite A-module where $A := \mathcal{O}_{\varphi_{|D|}(X), \varphi_{|D|}(P)}$. We need to show that it is A itself.

Base point freeness of $|D-P|$ implies that there exists $D'' \subseteq |D-P|$ such that $P \notin \operatorname{Supp}(D'')$. Moreover, by Proposition 8.29 $D'' = \varphi_{|D|}^*(H'')$ for a hyperplane H''. If g is a local equation for H'' around $\varphi_{|D|}(P)$ then $t = \varphi_{|D|}^* g$ satisfies $v_p(t) = 1$, i.e. t generates $M_P \subseteq \mathcal{O}_{X,P}$. So $A \subseteq \mathcal{O}_{X,P}$ such that $\mathcal{O}_{X,P}$ is finitely generated over A by the finiteness of $X \to \varphi_{|D|}(X)$. Also, $M_P = M\mathcal{O}_{X,P}$ where M is the maximal ideal of A corresponding to P. Moreover, $\mathcal{O}_{X,P}/M\mathcal{O}_{X,P} \cong A/M \cong \Bbbk$. Take an $f \in A$ such that $f \equiv 1$ modulo M. Then f modulo M generates $\mathcal{O}_{X,P}/M\mathcal{O}_{X,P}$ over A hence – by Nakayama's lemma – f generates $\mathcal{O}_{X,P}$ over A i.e. $A = \mathcal{O}_{X,P}$. □

Remark 8.36. In fact, the idea of the proof was that the morphism is injective and an isomorphism on all the cotangent spaces hence it is an isomorphism.

Ninth Lecture, 10th of November

8.5 Differential forms

Definition 8.37. Let X be a smooth quasi-projective variety, $p \in X$ and f a regular function in a neighborhood of P. Then we may associate an element $f - f(P) \in M_P \subseteq \mathcal{O}_{X,P}$. The induced element $\mathrm{d}_P(f)$ in $M_P/M_P^2 \cong T_P(X)^*$ is called the *differential* of f at P.

Lemma 8.38. *If $t_1, \ldots, t_n \in \mathcal{O}_{X,P}$ map to a $\Bbbk$-basis of M_P/M_P^2 then there exists an open neighborhood U of P: $t_1 - t_1(P), \ldots, t_n - t_n(P)$ map to a basis of M_Q/M_Q^2 for all $Q \in U$.*

Proof. [I don't understand this argument. Maybe I missed some indices?] We may assume that $X \subseteq \mathbb{A}^m$ is affine and $P = (0, \ldots, 0)$. We may also assume that $t_1, \ldots, t_n$ are some of the coordinate functions on $\mathbb{A}^m$. If $X = V(f_1, \ldots, f_r)$ then we have

$$\sum_{i=1}^{m} \partial_i f_j(P) t_i = 0$$

for all $j = 1, \ldots, r$. Assume that U is an open neighborhood such that $\partial_i f_j$ are regular and $\neq 0$ on U. So we can then express $t_i - t_i(Q)$ for all $i > n$ from $t_1 - t_1(Q), \ldots, t_n - t_n(Q)$. □

Let $\mathrm{d}f$ be a function which maps $P \mapsto \mathrm{d}_P(f) \in M_P$. By the lemma, $\mathrm{d}_P(t_1), \dots, \mathrm{d}_P(t_n)$ generate $T_Q(X)^*$ freely in an open neighborhood of P.

Definition 8.39. A *regular differential 1-form* on X is a map $\omega : P \mapsto \omega(P) \in T_P(X)$ such that for all $P \in X$ has an open neighborhood U where $\omega(P) = \sum f_i \mathrm{d}_P t_i$ where $t_1, \dots, t_n$ are as in the Lemma and $f_i \in \mathcal{O}(U)$.

These form a $\Bbbk$-vector space, denoted by Ω^1_X.

Proposition 8.40. *Every $P \in X$ has an open neighborhood U such that Ω^1_U is the free $\mathcal{O}(U)$-module on $\mathrm{d}t_1, \dots, \mathrm{d}t_n$.*

Remark 8.41. By definition, $\mathrm{d}t_1, \dots, \mathrm{d}t_n$ generate Ω^1_U as an $\mathcal{O}(U)$-module. If there is a relation $\sum g_i \mathrm{d}t_i = 0$ for some $g_i \in \mathcal{O}(U)$ on $U \backslash \operatorname{supp}\bigl(\operatorname{div}(g_i)\bigr)$ then we get a relation $\sum g_i(P)\mathrm{d}t_i = 0$ nontrivially.

Definition 8.42. For $p \geq 1$ a *regular p-form* on X is a map $P \mapsto \omega(P) \in \Lambda^p T_P(X)^*$ such that

$$\omega = \sum f_{i_1,\dots,i_p} \mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_p}$$

on a sufficiently small nonempty open U around P, $f_{i_1}, \dots, f_{i_p} \in \mathcal{O}(U)$.

Definition 8.43. A *rational* 1*-form* is represented by a pair (U, ω_U) where $U \subseteq X$ is a nonempty open, $\omega_U \in \Omega^1_U$. Two such pairs (U, ω_U), (V, ω_V) define the same rational 1-form if on $U \cap V$ we have $\omega_U|_{U\cap V} = \omega_V|_{U \cap V}$. Similarly, for p-forms.

Again, these form a $\Bbbk$-vector space $\Omega^1_{K(X)}$ and $\Omega^p_{K(X)}$ respectively. Moreover, $\Omega^1_{K(X)}$ has a natural $K(X)$-vector space structure (and similarly for $\Omega^p_{K(X)}$). Indeed, given an $\omega \in \Omega^1_{K(X)}$ and $f \in K(X)$ we represent ω by (U, ω_U) and we choose $U' \subseteq U$ so that $f \in \mathcal{O}(U')$. On U' we have $\omega_U = \sum f_i \mathrm{d}t_i$. Define $f\omega \in \Omega^1_{K(X)}$ as the class of

$$\Bigl(U', \sum f \cdot f_i \mathrm{d}t_i\Bigr)$$

One can check that it is well defined. Similarly, for p-forms.

Proposition 8.44. $\dim_{K(X)} \Omega^p_{K(X)} = \binom{n}{p}$ *where* $n = \dim X$*.*

Corollary 8.45. $\dim_{K(X)} \Omega^n_{K(X)} = 1$.

Proof. Fix $V \subseteq X$ a nonempty open such that Ω^1_V is freely generated by $\mathrm{d}t_1, \dots, \mathrm{d}t_n$ on V. If $\omega \in \Omega^p_{K(X)}$ is represented by (U, ω_U) then we may write

$$\omega_U = \sum f_{i_1,\dots,i_p} \mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_p}$$

on $U \cap V$ (which is dense open by irreducibility), hence $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$ give a $K(X)$-basis of $\Omega^p_{K(X)}$. ☐

Definition 8.46. Construction: Given a top differential form i.e. $\omega \in \Omega^n_{K(X)}$, we may associate a divisor $\operatorname{div}(\omega)$ such that its class in the Picard group is independent of ω. It will be called the *canonical class* which is now an invariant of the variety itself. It is typically denoted by K.

Construction: The given ω can be represented by (U, ω_U) for some nonempty open set U and $\omega_U \in \Omega^n_U$. Then for any $p \in X$ we may find an open neighborhood $U_P$of P such that $\Omega^n_{U_P}$ is a free $\mathcal{O}(U_P)$-module of rank 1 generated by $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$. Then we may express $\omega_U = f_p \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$ on $U \cap U_P$ for some $f_p \in \mathcal{O}(U \cap U_P) \subseteq K(X)$.

We claim that the system (U_P, f_P) defines a Cartier divisor on X. If Q is another point on U_Q then $\Omega^n_{U_Q}$ is generated by an element $\mathrm{d}s_1 \wedge \ldots \mathrm{d}s_n$. On $U_P \cap U_Q$ we must have

$$\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n = \mathrm{Jac}\begin{pmatrix} s_1 & \ldots & s_n \\ t_1 & \ldots & t_n \end{pmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$$

since $\Omega^n_{K(X)}$ is one dimensional and the scalar factor between the two generators is the Jacobian of the "transition function" between the coordinate system $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$. Observe that this Jacobian is invertible on $\mathcal{O}(U_P \cap U_Q)$ hence (U_P, f_P) is indeed a Cartier divisor.

Proposition 8.47. *The class* $[K]$ *is independent of* ω.

Proof. By the Corollary 8.45, $\Omega^n_{K(X)}$ is one dimensional hence any other ω is a scalar multiple of the original one. So their difference is a divisor of a constant function which is regular. □

Definition 8.48. The associated linear system $|K|$ of K is called the canonical linear system. It is again canonically attached to the variety, independent of ω.

Theorem 8.49. (Riemann Roch Theorem for curves) *Let* X *be a smooth projective curve,* $D \in \mathrm{Div}(X)$ *and* K *a canonical divisor. Then*

$$\dim_{\Bbbk} \mathcal{L}(D) - \dim_{\Bbbk} \mathcal{L}(K - D) = \deg D - g + 1$$

where $g := \dim \mathcal{L}(K)$ *called the* genus *of the curve.*

Remark 8.50. Note that $\mathcal{L}(K) := \{f \in K(X)^\times \mid \mathrm{div}(f) + K \geq 0\}$ where now $K = \mathrm{div}(\omega)$ hence our assumption on f is that $\mathrm{div}(f) + \mathrm{div}(\omega) = \mathrm{div}(f\omega) \geq 0$ by additivity of div(.) [How do we know that for ω?]. By this one can get $g = \dim_{\Bbbk} \mathcal{L}(K) = \dim_{\Bbbk} \Omega^1_X$.

Corollary 8.51.

1. *Apply the theorem for* $D = K$*, we get that* $\deg K = 2g - 2$.

2. *The previous version, Theorem 8.25 follows by Theorem 8.49: If* $\deg D < 0$ *then* $\mathcal{L}(D) = 0$*. (Indeed, if* $f \in \mathcal{L}(D)\backslash\{0\}$ *and we take* $D' = D + \mathrm{div}(f) \geq 0$ *then they are the same in the Picard group hence* $\deg D' = \deg D < 0$ *but an effective divisor* D' *cannot have negative degree. So if* $\deg D > 2g - 2$ *then* $\mathcal{L}(K - D) = 0$ *and so* $\dim \mathcal{L}(D) = \deg D - g + 1$.

Example 8.52. If $X = \mathbb{P}^1$, $U_+ = \mathbb{A}^1$, $U_- = \mathbb{P}^1\backslash\{0\}$ then $\mathcal{A}_{U_+} = \Bbbk[t]$ and $\mathcal{A}_{U_-} = \Bbbk[t^{-1}]$. Let us compute $\mathrm{div}(\mathrm{d}t)$: on U_+ it is given by the function 1 and on U_- it is given by $-\frac{1}{t^2}$ since $\mathrm{d}(t^{-1}) = -\frac{1}{t^2}\mathrm{d}t$. Hence, $\deg K = -2$ so $\Omega^1_{\mathbb{P}^1} = 0$ because every differential form must have a pole somewhere. For $D \geq 0$ Riemann Roch gives $\mathcal{L}(D) = \deg D + 1$. (It can be computed explicitly too, optional homework)

Remark 8.53. Note that the multiples of the canonical class and the canonical linear system are also invariants of the curve. So consider the linear system $|mK|$ for $m \geq 1$. If it is base point free then they define a morphism $X \to \mathbb{P}^{g-1}$ for $m = 1$. If $g > 2$ and $m > 1$ then $\deg mK = m(2g - 2) > 2g$. Hence we get an embedding $\varphi_{|mK|} : X \hookrightarrow \mathbb{P}^{m(2g-2)-g}$. The same works for $g = 2$ and $m = 3$. So we get that if $g > 1$ then $\varphi_{|3K|}$ embeds X into $\mathbb{P}^{5g-6}$ as a curve of degree $6g - 6$.

Remark 8.54. For an arbitrary variety X, an important invariant of X is the *Kodaira dimension*:

$$\kappa(X) := \max \dim_{\Bbbk} \left\{\overline{\varphi_{|mK|}} \mid m \geq 1\right\} \leq \dim X$$

Even so $\varphi_{|mK|}(X)$ may be Φ. In this case, set $\kappa(X) = -1$ (or $-\infty$ depending on the conventions). Varieties of maximal Kodaira dimension are called *varieties of general type*. Curves of genus at least 2 have Kodaira dimension 1. Curves of genus one have Kodaira dimension 0 and the ones of genus 1 are of Kodaira dimension -1. (For a proof, see Wikipedia)

Proposition 8.55. *Assume that X is a smooth projective curve of genus at least 2. Then either $\varphi_{|K|} : X \hookrightarrow \mathbb{P}^{g-1}$ is an isomorphic embedding or $\varphi_{|K|} : X \to \mathbb{P}^1$ such that $\big[K(X) : K(\mathbb{P}^1)\big] = 2$. The curves in the latter class are called hyperelliptic curves.*

Proof. Optional homework. The idea is the same as in Theorem 8.31 last time. We prove it next time. □

Assumptions: Let $\varphi : X_1 \to X_2$ be a finite surjective morphism of smooth projective curves. Then φ induces a field extension $K(X_1) \mid K(X_2)$. Assume, moreover, that it is separable.

We have seen that φ induces a pullback map on rational forms:

$$\varphi^* : \Omega^1_{K(X_2)} \to \Omega^1_{K(X_1)} \qquad f\mathrm{d}t \mapsto \varphi^*(f)\mathrm{d}(\varphi^* t)$$

Fact 8.56. *If $K(X_1) \mid K(X_2)$ is separable then* $\ker(\varphi^*) = 0$.

Remark 8.57. In characteristic p it may happen that $\varphi^* t = s^p$ so $\mathrm{d}(\varphi^* t) = p \cdot s^{p-1}\mathrm{d}s = 0$. The fact says that this is the only obstruction to the injectivity of φ^*.

Observation: If $\omega \in \Omega^1_{X_2}$ is regular then $\varphi^*\omega \in \Omega^1_{X_1}$ is regular.

Corollary 8.58. *If $X_1 = \mathbb{P}^1$ then $X_2 \cong \mathbb{P}^1$.*

Proof. If $g(X_2) > 0$ then there is a nonzero regular differential form $\omega \in \Omega^1_{X_2}$ then $\varphi^*\omega \in \Omega^1_{\mathbb{P}^1} = 0$ so $\ker(\varphi^*) \neq 0$. That is a contradiction. □

Translation: (Lüroth's Theorem) If $\mathbb{k}$ is an algebraically closed field, and we have the extensions $\mathbb{k}(t) \supseteq L \supseteq \mathbb{k}$ where $|\mathbb{k}(t) : L| < \infty$ and $\mathbb{k}(t) \mid L$ is separable then $L = \mathbb{k}(t)$.

Definition 8.59. Let $Q \in X_1$, $P = \varphi(Q) \in X_2$, t a generator of $M_p \subseteq \mathcal{O}_{X_1,P}$ and s a generator of $M_Q \subseteq \mathcal{O}_{X,P}$. Then $\varphi^* t = u \cdot s^{e_Q}$ where u is a unit in $\mathcal{O}_{X_1,Q}$ and $e_Q > 0$. We call e_Q the *ramification index* of φ at Q and we say that φ is *ramified* at Q if $e_Q > 1$.

Remark 8.60. e_Q does not depend on s and t.

Proposition 8.61. $\sum_{Q\in\varphi^{-1}(P)} e_Q = \big[K(X_1) : K(X_2)\big]$.

Proof. Choose an affine subset $U \subseteq X_2$ containing P such that $\varphi^{-1}(U) = V$ is also affine and $\mathcal{A}_V$ is finitely generated as an $\mathcal{A}_U$-module. Then $\mathcal{O}_{X_2,P} = \big(\mathcal{A}_U\big)_{M_P}$ and all the points $Q \in \varphi^{-1}(P)$ correspond to the maximal ideals of $B := \big(\mathcal{A}_V\big)_{M_P\mathcal{A}_V}$. If $M_P = (t)$ then $tB \supseteq M_Q^n$ for high enough n and

$$B/tB \cong \prod_{Q\in\varphi^{-1}(P)} B_{M_Q}\Big/tB_{M_Q}$$

(see, the proof of Theorem 8.17). By $\dim_{\mathbb{k}} B_{M_Q}/tB_{M_Q} = e_Q$ we get that $\dim_{\mathbb{k}} B/tB = \sum_Q e_Q$. However, since $\mathcal{O}_{X_2,P}$ is a principal ideal domain, B is a finitely generated free module because it is finitely generated and torsion free. So the rank of B over A is $\dim_{\mathbb{k}} B/tB$. On the other hand, we also have $\mathrm{rk}_A B = \big[K(X_1) : K(X_2)\big]$ since $K(X_1) \supseteq B$ and $K(X_2) \supseteq \mathcal{O}_{X,P}$. □

Corollary 8.62. *If $D \in \mathrm{Div}(X_2)$ then* $\deg\big(\varphi^* D\big) = \big[K(X_1) : K(X_2)\big] \cdot \deg D$.

Proposition 8.63. *Assume that $\omega \in \Omega^1_{K(X_2)}$ and that* $\mathrm{char}\mathbb{k} \nmid e_Q$ *for all $Q \in X_1$. Then*

$$\mathrm{div}\big(\varphi^*\omega\big) = \varphi^*\mathrm{div}(\omega) + \sum_{Q\in X_1} (e_Q - 1)Q$$

Corollary 8.64. *By the Proposition, we get that $e_Q > 1$ for only finitely many Q since $\operatorname{div}\big(\varphi^*\omega\big) - \varphi^*\operatorname{div}(\omega)$ is a sum of finitely many points (with multiplicity).*

Remark 8.65. The separability assumption is crucial here, in the corollary, and also in the next theorem. (If the extension is inseparable then almost all points ramify.)

Corollary 8.66. (Hurwitz genus formula) *Let $g(X_i)$ be the genus of X_i for $i = 1, 2$. Then*

$$2g(X_1) - 2 = \big[K(X_1) : K(X_2)\big] \cdot \big(2g(X_2) - 2\big) + \sum_{Q \in X_1} e_Q - 1$$

In particular, $g(X_1) \geq g(X_2)$. [I don't see how it follows from the previous propositions.]

Proof of Proposition 8.63. Let t be a generator of $M_P \subseteq \mathcal{O}_{X_2,P}$ and s a generator of $M_Q \in \mathcal{O}_{X_1,Q}$. Then $\varphi^* t = u \cdot s^{e_Q}$ for some u and e_Q. Moreover, $\omega = f \cdot \mathrm{d}t$ in a neighborhood of P for some $f \in K(X_1)$. It means that

$$\varphi^*\omega = (\varphi^* f)\mathrm{d}(\varphi^* t)$$

Hence, $\operatorname{div}(\varphi^*\omega) = \operatorname{div}(\varphi^* f) + \operatorname{div}(\mathrm{d}\varphi^* t)$ where we can already compute these terms:

$$\mathrm{d}\varphi^* t = \mathrm{d}(u \cdot s^{e_Q}) = s^{e_Q}\mathrm{d}u + e_Q s^{e_Q - 1}\mathrm{d}s =$$

However, $\mathrm{d}u = g \cdot \mathrm{d}s$ for some $g \in \mathcal{O}_{X_1,Q}$. Hence,

$$= s^{e_Q - 1}(g \cdot s + e_Q)\mathrm{d}s$$

where e_Q is nonzero by the assumption $\operatorname{char}\mathbb{k} \not| e_Q$ so $g \cdot s + e_Q$ is a unit in $\mathcal{O}_{X_1,Q}$. Now, to get the claim, we compute both sides pointwise for Q:

$$v_Q\big(\operatorname{div}(\varphi^*\omega)\big) = v_Q\big(\operatorname{div}(\varphi^* f)\big) + e_Q - 1 =$$

since $v_Q\big(\operatorname{div}(\mathrm{d}\varphi^* t)\big) = v_Q\big(s^{e_Q-1}(g \cdot s + e_Q)\mathrm{d}s\big) = v_Q\big(s^{e_Q - 1}\mathrm{d}s\big) = e_Q - 1$ by the previous observations. So

$$= v_Q\big(\varphi^*\operatorname{div}(f)\big) + e_Q - 1 = v_Q\big(\varphi^*\operatorname{div}(\omega)\big) + e_Q - 1$$

as $\omega = f\mathrm{d}t$ hence, after summing the equations for Q we get the statement. ☐

Tenth Lecture, 17th of November

Theorem 8.67. *If $g \geq 2$ then $\varphi_{|K|} : X \to \mathbb{P}^{g-1}$ is either an isomorphic embedding or it induces a map $X \to \mathbb{P}^1$ such that $\big[K(X) : K(\mathbb{P}^1)\big] = 2$. (In the latter case, X is called hyperelliptic.)*

Proof. One has to check whether $|K|$ and $|K - P|$ are base point free for all $P \in X$. First, $|K|$ is always base point free (hence $\varphi_{|K|}$ is an embedding). We claim that for any $P \in X$, we have $\dim \mathcal{L}(K - P) = \dim \mathcal{L}(K) - 1 = g - 1$. Indeed, by the Riemann Roch we know that $\dim \mathcal{L}(P) - \dim \mathcal{L}(K - P) = 1 - g + 1$, so we only have to prove that $\dim \mathcal{L}(P) = 1$.

Assume indirectly, that $\dim \mathcal{L}(P) > 1$. It means that there exists a nonconstant $f \in \mathcal{L}(P)$ with a pole of order at most 1 at P and no poles elsewhere. Hence, it gives that $\varphi_f : X \to \mathbb{P}^1$ where $\big[K(X) : K(\mathbb{P}^1)\big] = 1$ i.e. $X \cong \mathbb{P}^1$. But it contradicts $g \geq 2$.

Next, we have to find out when is $|K - P|$ base point free. It is base point free if and only if

$$\dim \mathcal{L}(K - P - Q) = \dim \mathcal{L}(K - P) - 1 = \dim \mathcal{L}(K) - 2 = g - 2$$

By Riemann-Roch, we get that

$$\dim \mathcal{L}(P + Q) - \dim \mathcal{L}(K - P - Q) = \deg(P + Q) - g + 1 = 3 - g$$

So we win if $\dim \mathcal{L}(P + Q) = 1$. If it does not hold then there exists a nonconstant function $f \in \mathcal{L}(P + Q)$ such that $\varphi_f : X \to \mathbb{P}^1$ where $\big[K(X) : K(\mathbb{P}^1)\big] = \deg(P + Q) = 2$. ☐

Remark 8.68. If $g = 2$ then X is hyperelliptic. Indeed, then $\deg K = 2g - 2 = 2$ hence $\varphi_{|K|} : X \to \mathbb{P}^1$ is of degree 2.

Theorem 8.69. (Hurwitz genus formula) *Let $\varphi : X_1 \to X_2$ be a surjective morphism of smooth projective curves. Then the following holds:*

$$2g(X_1) - 2 = \big[K(X_1) : K(X_2)\big] \cdot \big(2g(X_2) - 2\big) + \sum_P e_P - 1$$

Remark 8.70.

1. It is Hurwitz's theorem that if $g \geq 2$ then $|\mathrm{Aut}(X)| < \infty$. In fact, if $\mathrm{char}(\Bbbk) = 0$ then $|\mathrm{Aut}(X)| \leq 84(g-1)$. The latter is false in positive characteristic. For the proof, see Hartshorne.

2. If $\Bbbk = \mathbb{C}$ then X has the structure of a Riemann surface: by a generator of $\mathcal{O}_{X,P}$ one can give a bijection between an open subset of $\mathbb{C}$ and a neighborhood of P. Moreover, X is compact as a meromorphic function (i.e. $X \to \mathbb{P}^1$) is proper hence the inverse image is compact. It is a topological fact that X must be homeomorphic to one of the canonical orientable surfaces that are connected sums of tori with g_{top} holes. We also have $\chi(X) = 2 - 2g_{\mathrm{top}}$. Also, in this case a morphism $X \to Y$ gives rise to a holomorphic map of Riemann surfaces, locally at a point P it looks like $z \mapsto z^e$ for some $e \in \mathbb{Z}_{>0}$. In fact, this e is the same as the algebraic multiplicity e_P at P. For these quantities, we have a topological Hurwitz formula:

   $$\chi(X) = \big[K(X) : K(Y)\big] \cdot \chi(Y) - \sum_P (e_P - 1)$$

   It follows from pulling back a triangulation of Y for X and investigating the inverse image of edges, faces and most importantly, vertices. As a corollary, we get that

   $$2g_{\mathrm{top}}(X) - 2 = \big[K(X) : K(Y)\big] \cdot (2g_{\mathrm{top}}(Y) - 2) - \sum_P (e_P - 1)$$

   Therefore, $g_{\mathrm{top}}(X) = \mathrm{g}(X)$ since $g_{\mathrm{top}}(\mathbb{P}^1) = g(\mathbb{P}^1) = 0$ and now we can apply both formulas with $Y = \mathbb{P}^1$.

9 Blowups

Definition 9.1. *(Blowup of $\mathbb{A}^n$ at the origin O)* Consider the canonical projection $p : \mathbb{A}^n \backslash \{0\} \to \mathbb{P}^{n-1}$. The graph of p is $\Gamma_p \subseteq \big(\mathbb{A}^n \backslash \{0\}\big) \times \mathbb{P}^{n-1}$. Then the blowup $B_O(\mathbb{A}^n)$ of O in $\mathbb{A}^n$ is the Zariski closure of Γ_p inside $\mathbb{A}^n \times \mathbb{P}^{n-1}$. It comes with two natural projections such that the following diagram commutes

$$\begin{array}{ccccc}
\Gamma_p & \hookrightarrow & B_O(\mathbb{A}^n) & \hookrightarrow & \mathbb{A}^n \times \mathbb{P}^{n-1} \\
\downarrow & & \downarrow{\scriptstyle p_1} & \searrow^{p_2} & \\
\mathbb{A}^n \backslash \{0\} & \hookrightarrow & \mathbb{A}^n & & \mathbb{P}^{n-1}
\end{array}$$

Note that $p_1 : B_O(\mathbb{A}^n) \to \mathbb{A}^n$ is a birational morphism because it is an isomorphism above $\mathbb{A}^n \backslash \{0\}$.

Question: What is $p_1^{-1}(0)$?

Take a line $L \subseteq \mathbb{A}^n$ through O and consider its image $P_L := p(L \backslash \{0\}) \in \mathbb{P}^{n-1}$. Then L decomposes as $L = \{0\} \cup p^{-1}(P_L)$ hence $(L \backslash \{0\}) \times \{P_L\} \subseteq \Gamma_p$. By the definition of the blowup, it means

$$L \times \{P_L\} \subseteq B_O(\mathbb{A}^n) = \overline{\Gamma_p}$$

It holds for all line L in $\mathbb{A}^n$ hence $\{0\} \times \mathbb{P}^{n-1} \subseteq B_0(\mathbb{A}^n)$. So the answer is that the inverse image of 0 is isomorphic to $\mathbb{P}^{n-1}$.

Remark 9.2. $B_O(\mathbb{A}^n)$ is irreducible hence it is a quasi-projective variety. Indeed, $\Gamma_p \cong \mathbb{A}^n\backslash\{0\}$ is irreducible so its closure too.

Equations for the blowup as a closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ where we can describe the subsets as zero locus of polynomials in $2n$ variables that are homogeneous in the last n variables. Let us denote the variables by $x_0, \dots, x_{n-1}$ on $\mathbb{A}^n$ and $y_0, \dots, y_{n-1}$ on $\mathbb{P}^{n-1}$. Now, on a coordinate chart $\mathbb{A}^n \times D_+(y_i)$ the coordinates are $x_0, \dots, x_{n-1}, \frac{y_0}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_n}{y_i}$. Then the equations for $\Gamma_p \cap \big(\mathbb{A}^n \times D_+(y_i)\big)$ is

$$\frac{x_j}{x_i} = \frac{y_j}{y_i}$$

for all $j = 0, \dots, i-1, i+1, \dots, n-1$. Hence, its closure is defined by the equations $x_j = \frac{y_j}{y_i} x_i$ for all $j = 0, \dots, i-1, i+1, \dots, n-1$. These patch together as the locus

$$V(x_i y_j - x_j y_i \mid \forall i, j) \subseteq \mathbb{A}^n \times \mathbb{P}^n$$

Note that these are 2×2 subdeterminants of a $2 \times n$ matrix filled with the variables x and y.

It gives the motivation for the next definition.

Definition 9.3. *Blowup of* $\mathbb{P}^n$ at $(1, 0, \dots, 0)$. Then

$$B_O(\mathbb{P}^n) = V(x_i y_j - x_j y_i \mid 1 \leq i, j \leq n) \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$$

where $x_0, \dots, x_{n-1}$ are coordinate on $\mathbb{P}^n$ and $y_1, \dots, y_n$ are coordinate on $\mathbb{P}^{n-1}$ and x_0 is purposefully left out in the definition.

Remark 9.4. Notice that $B_O(\mathbb{P}^n) \cap \big(D_+(x_0) \times \mathbb{P}^{n-1}\big) \cong B_O(\mathbb{A}^n)$ while $B_O(\mathbb{P}^n) \cap \big(D_+(x_i) \times \mathbb{P}^{n-1}\big) \cong \mathbb{A}^n$ for all $i > 0$.

Definition 9.5. *(General definition of blowup)* Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety. Assume that $O = (1, 0, \dots, 0) \in X$. Then $B_O(X)$ is defined as the closure of $p_1^{-1}\big(X\backslash\{0\}\big)$ in $B_O(\mathbb{P}^n)$.

It has 2 components: one of $p_1^{-1}(0) \cong \mathbb{P}^n \cap B_O(X)$ called the *exceptional divisor* and the other component $\tilde{X}$ that is called the *proper transform* of X.

Example 9.6. For an odd integer $n > 1$, consider $X_n = V(x_0^2 - x_1^n) \subseteq \mathbb{A}^2$ has a singular point at $(0, 0)$. Now, we blow it up at $(0, 0)$: then $B_O(\mathbb{A}^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ has an open covering by 2 copies of $\mathbb{A}^2$. Let y_0, y_1 be the coordinates of $\mathbb{P}^1$. Consider, $U_1 = B_O(\mathbb{A}^2) \cap \big(\mathbb{A}^2 \times D_+(y_1)\big)$ and $U_0 = B_O(\mathbb{A}^2) \cap \big(\mathbb{A}^2 \times D_+(y_0)\big)$. Turning to the coordinate rings of these, we get

$$\mathcal{A}_{U_1} = \Bbbk\Big[x_0, x_1, \frac{y_0}{y_1}\Big] / \Big(x_0 - \frac{y_0}{y_1} x_1\Big) \cong \Bbbk\Big[x_1, \frac{y_0}{y_1}\Big]$$

$$\mathcal{A}_{U_0} = \Bbbk\Big[x_0, x_1, \frac{y_1}{y_0}\Big] / \Big(x_1 - \frac{y_1}{y_0} x_0\Big) \cong \Bbbk\Big[x_0, \frac{y_1}{y_0}\Big]$$

So the exceptional divisors are $p_1^{-1}(0) \cap U_1 = V(x_1)$ and $p_1^{-1}(0) \cap U_0 = V(x_0)$. Hence, $p_1^{-1}(X_n\backslash\{0\}) \subseteq U_0 \cap U_1$. Explicitly, on U_1 the equation of $p_1^{-1}(X_n\backslash\{0\})$ is:

$$\Big(\frac{y_0}{y_1} x_1\Big)^2 = x_1^n$$

where we can simplify by x_1 giving $\Big(\frac{y_0}{y_1}\Big)^2 = x_1^{n-2}$. Similarly, on U_0. The conclusion is that $p_1^{-1}(X_n) \cong \mathbb{P}^1 \cup X_{n-2}$.

Remark 9.7. As the example shows, blow up is a constructive way to (iteratively) resolve the singularities of curves.

9.1 Properties of the blowup

Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety, $p \in X$ is a smooth point, $\pi : B_p(X) \to X$. We choose coordinates in a way that $D_+(x_0) \ni p$ and we denote the local coordinates on $D_+(x_0)$ by $x_1, \dots, x_n$. We may assume that $x_1, \dots, x_d$ give a basis of M_p/M_p^2 where M_p is the maximal ideal of $\mathcal{O}_{X,p}$.

Proposition 9.8. *There exists an open neighborhood U of p, a variety W_X and an isomorphism between a subset of the blowup and a subset of W_X*

$$\begin{array}{ccccccc} B_p(X) & \hookleftarrow & \pi^{-1}(X \cap U) & \cong & \pi_X^{-1}(X \cap U) & \hookrightarrow & W_X \\ & & & \searrow \quad \swarrow_{\pi_X} & & & \\ & & & X \cap U & & & \end{array}$$

in a way that $X \times \mathbb{P}^{d-1} \supseteq W_X = V(w_i y_j - w_j y_i \mid 1 \le i, j \le d)$ where $y_1, \dots, y_d$ are the coordinates of $\mathbb{P}^{d-1}$ and $w_0, \dots, w_n$ are some well chosen coordinates of the projective space $\mathbb{P}^n$ which contains X .

Proof. We may assume that the first d basis vectors (in $D_+(x_0)$) after $x_1, \dots, x_d$ give a basis of the cotangent space M_p/M_p^2. Now, we extend this by other coordinates $x_1, \dots, x_d, w_{d+1}, \dots, w_n$ such that they give a basis modulo $M_{\mathbb{P}^n,p}^2$ and there exists an open neighborhood U of p such that $X \cap U = V(w_{d+1}, \dots, w_n) \cap U$.

Lemma 9.9. *$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_d, w_{d+1}, \dots, w_n)$ induces an isomorphism of a neighborhood U of p in X with a neighborhood U' of $O \in \mathbb{A}^n$.*

Proof. In $\mathcal{O}_{\mathbb{P}^n,p}$ we can write $w_j = \sum a_{ij} x_i$ for some $a_{ij} \in \mathcal{O}_{\mathbb{P}^n,p}$ such that $\det((a_{ij}(p))) \neq 0$ modulo $M_{\mathbb{P}^n,p}^2$ hence it has an inverse matrix in $\mathcal{O}_{\mathbb{P}^n,p}$ which gives an isomorphism on some open neighborhood. □

If $W \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ is defined by $V(w_i y_j - w_j x_i)$ using this new coordinate system then we get an isomorphism between the open subsets U and U':

$$\begin{array}{ccccccc} B_p(\mathbb{P}^n) & \hookleftarrow & \pi^{-1}(U) & \xrightarrow{\cong} & \pi_W^{-1}(U') & \hookrightarrow & W \\ & \searrow & \downarrow & & \downarrow & \swarrow_{\pi_W} & \\ & & U & \xrightarrow{\cong} & U' & & \end{array}$$

So we may identify $X \cap U$ with its image in U'. □

Assumptions: From now on X is a smooth quasi-projective variety such that $\dim(X) = 2$ and $p \in X$.

Definition 9.10. If $D \in \mathrm{Div}(X)$ and $p \in X$ the the multiplication of D at p is defined as

$$\mu_p(D) := \max\{r \mid f \in M_p^r\}$$

where $D = \mathrm{div}(f)$ in a neighborhood of P.

Remark 9.11. We have $\mu_p(D) = 0$ if and only if $p \notin \mathrm{Supp}(D)$ and $\mu_p(D) = 1$ if and only if p is a smooth point on $\mathrm{Supp}(D)$. Indeed, $\mu_p(D) = 1$ means that

$$1 = \dim M_{D,p}/M_{D,p}^2 = \dim \left(M_p/M_p^2 + (f)\right)$$

which is equivalent to saying that $\dim M_p/M_p^2 = 2$ i.e. that p is smooth.

Lemma 9.12. *Consider $\pi_p : B_p(X) \to X$. Then we have*

$$\pi_p^* D := \tilde{D} + \mu_p(D)E$$

where $\tilde{D}$ is a the strict transform of D and E is the exceptional divisor (i.e. $E \cong \mathbb{P}^1$)

Proof. We can assume that D has p with nonnegative coefficient, else we interchange it with $-D$. Clearly, we have $\operatorname{Supp}\big(\pi_p^*(D) - \tilde{D}\big) = E$ by definition of $\tilde{D}$. Choose an affine open neighborhood of P and local coordinates x, y on U such that $\pi_p^{-1}(U) \subseteq U \times \mathbb{P}^1$ is given by $xt_1 = yt_0$ where t_0 and t_1 are coordinates of $\mathbb{P}^1$. By Lemma 9.8, this is possible. Then $\pi_p^{-1}(U) \cap \big(U \times D_+(x_0)\big)$ has equation $y = x \cdot \frac{t_1}{t_0}$. Take $E = V(x)$. If we set $r := \mu_p(D)$ and U is a neighborhood of p such that D has a local equation f on U then this f is regular by our assumption on the coefficient of p in D.

The local equation has the form $f = \varphi(x, y) + \psi$ for some $\psi \in M_p^{r+1}$ and some $\varphi \in \mathcal{O}_{X,p}$ such that $\deg \varphi = r$. Then

$$\pi_p^*(f) = \pi_p^*\varphi + \pi_p^*\psi = \pi_p^*\varphi(x, xv) + \pi_p^*\psi(x, xv)$$

hence by $E = V(x)$ we get $\pi_p^*(D) - \tilde{D} = rE$. □

9.2 Intersection pairing of the surface X

Definition 9.13. Let $C_1, C_2 \subseteq X$ are irreducible curves with $I_1, I_2 \subseteq \mathcal{O}_{X,p}$ the corresponding ideals for some fixed $p \in X$. Then

$$i_p(C_1, C_2) = \dim \mathcal{O}_{X,p}/(I_1 + I_2)$$

Lemma 9.14.

1. *$i_p(C_1, C_2) = i_p(C_2, C_1)$*
2. *Let us denote the inclusion map $\rho : C_2 \hookrightarrow X$. Then $i_p(C_1, C_2) = m_p(\rho^* C_1)$ on C_2.*

Theorem 9.15. *For a smooth projective surface X there exists a unique symmetric $\mathbb{Z}$-bilinear pairing*

$$(.,.) \, : \, \operatorname{Div}(X) \times \operatorname{Div}(X) \to \mathbb{Z}$$

such that

- *for any two irreducible curves $C_1 \neq C_2 \subseteq X$ we have*

$$(C_1, C_2) = \sum_{p \in X} i_p(C_1, C_2)$$

- *for any $f \in K(X)^\times$ and $D \in \operatorname{Div}(X)$ we have $(\operatorname{div}(f), D) = 0$. In particular, it gives a well-defined pairing on the Picard group too.*

Lemma 9.16. (Moving lemma) *Assume that $D \in \operatorname{Div}(X)$ and $p_1, \ldots, p_m$ are points on X. Then there exists a divisor $D' \in \operatorname{Div}(X)$ such that $[D] = [D'] \in \operatorname{Pic}(X)$ and $p_1, \ldots, p_m$ are not in $\operatorname{Supp}(D')$.*

Remark 9.17. The lemma is true in higher dimension too with the appropriate modifications and it can be proved the same way as we will prove it. Since we will not need it in that generality, we stick to the terminology of surfaces.

Proof. of Theorem 9.15: It is enough to define (C, D) for an irreducible curve $C \subseteq X$ and $D \in \mathrm{Div}(X)$ since then we can extend it $\mathbb{Z}$-linearly. By the Moving lemma 9.16 there exists a $D' \in \mathrm{Div}(X)$ such that $[D] = [D'] \in \mathrm{Pic}(X)$ and such that $C \not\subseteq \mathrm{Supp}(D')$. Indeed, choose a point from C: then if D' avoids it then it cannot contain C.

Now, define (by the first property in the theorem)

$$(C, D) := (C, D') = \sum_{p,i} m_i i_p(C, D'_i)$$

for the $D' = \sum_i m_i D'_i$ chosen before. (Note that we cannot assume that D' is irreducible since the Moving lemma does not necessarily give such a divisor.) This is well-defined because if D' and D'' are good enough (by the Moving lemma 9.16, it can be assumed) then $D' - D'' = \mathrm{div}(f)$ but

$$\big(C, \mathrm{div}(f)\big) = \sum_p m_p(f)|_C$$

which is zero by Theorem 8.17. In particular, the second property is also satisfied. □

Corollary 9.18. *Let $\pi_p : B_p(X) \to X$ be the blowup at p.*

1. *If $D_1, D_2 \in \mathrm{Div}(X)$ then $(D_1, D_2) = (\pi_p^* D_1, \pi_p^* D_2)$ (where the latter is well-defined since it is a union of a line, the exceptional divisor and an already smooth quasi-projective part, the proper component)*
2. *If $D \in \mathrm{Div}(X)$ then $(\pi_p^* D, E) = 0$*
3. *For the exceptional divisor, we have$(E, E) = -1$*
4. *If $\tilde{D}_1, \tilde{D}_2$ are the strict transforms of D_1 and D_2 then $(\tilde{D}_1, \tilde{D}_2) = (D_1, D_2) - \mu_p(D_1)\mu_p(D_2)$.*

Eleventh Lecture, 27th of November

Corollary 9.19 (Corollary of Theorem 9.15)**.** $(.,.)$ *induces a $\mathbb{Z}$-bilinear map* $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$.

Proof of Lemma 9.16. It is enough to treat the case where D is an irreducible curve since if $D = \sum m_i D_i$ and $D_i + \mathrm{div}(f_i)$ avoid $p_1, \ldots, p_m$ then their sum $D + \mathrm{div}(\prod f_i)$ also avoid $p_1, \ldots, p_m$. We prove by induction on n. For $n = 1$: if $p \in \mathrm{Supp}(D)$ then we can take a local equation f around P such that $\mathrm{div}(f)$ in this neighborhood defines D. Now, take the divisor $D - \mathrm{div}(f)$, then it already defines the empty divisor on a sufficiently small neighborhood of P. Hence, it is a good replacing divisor, although it may give extra components to D outside of the mentioned neighborhood of P.

Now, to prove the induction step, assume that the lemma is true for $m - 1$, moreover, assume that X is affine and $p_i \in X$ for all i. One can check that we may assume this. Then there exists an ideal $I \subseteq \mathcal{A}_X$ such that $\mathrm{Supp}(D) = V(I)$. Similarly, the p_i's correspond to maximal ideals $M_i \lhd \mathcal{A}_X$. By induction we may assume that $p_i \notin \mathrm{Supp}(D)$ for all $1 \leq i \leq m - 1$ i.e. there exist $g_i \in I \cap \big(\cap_{j \neq i} M_j\big)$ but $g_i \notin M_i$. Now, take a local equation of D around p_m. (If D is empty around p_m we won, so we may assume that this is not the case.)

We claim that we may choose $f \in \mathcal{A}_X$. Indirectly, assume that $\mathrm{div}_\infty(f) = \sum m_l D_l \neq 0$ where $p_m \notin D_l$ for all l since D was an effective divisor. Then, for all l there exists an $f_l \in \mathcal{A}_X$ such that $f_l(p_m) \neq 0$ but $D_l \subseteq V(f_l)$ by the Nullstellensatz. Now, take $f \cdot \prod_l f_l^{m_l}$ which is already an regular function and since $\prod f_l^{m_l}(p_m) \neq 0$ we get that it is a local equation of D around p_m. So we may replace f by $f \cdot \prod_l f_l^{m_l}$.

Now, by $g_i \notin M_i$ we get that there exist $\alpha_i \in \Bbbk^\times$ such that $g := f + \sum_{i=1}^{m-1} \alpha_i g_i^2 \notin M_i$ for all i. Indeed, we have to solve the equations $f(p_i) + \alpha_i g_i^2(p_i) \neq 0$ $(i = 1, \ldots, m-1)$ for $\alpha_1, \ldots, \alpha_{m-1}$ as other terms in the sum become zero at p_i. Therefore, we got $p_i \notin \mathrm{Supp}\big(D - \mathrm{div}(g)\big)$ for all $1 \leq i \leq m - 1$ since $p_i \notin \mathrm{Supp}\big(\mathrm{div}(g)\big)$ for all i. Moreover, we claim that g is a local equation for D at P_m: Indeed, in $\mathcal{O}_{X,p}$ which is a unique factorization domain, we get that $f \mid g_i$ by $g_i \in I$. Therefore, $f^2 \mid \sum_{i=1}^{m-1} \alpha_i g_i^2$ so $g = f \cdot u$ in $\mathcal{O}_{X,p}$ for a unit u. So, finally, we get $p_m \notin \mathrm{Supp}(D - \mathrm{div}(g)$ as we claimed. □

Proposition 9.20. *Let X be a smooth projective surface, $p \in X$ and $\pi_p : B_p(X) \to X$ the blow up of X at p. Then*

1. *If $D_1, D_2 \in \operatorname{Div}(X)$ then $\left(\pi_p^* D_1, \pi_p^* D_2\right) = (D_1, D_2)$.*
2. *If $D_1, D_2 \in \operatorname{Div}(X)$ and E is the exceptional divisor $\pi_p^{-1}(p)$ then $(\pi_p^* D, E) = 0$.*
3. *For the exceptional divisor, we have $(E, E) = -1$.*
4. *If $D_1, D_2 \in \operatorname{Div}(X)$ with strict transforms $\tilde{D}_1, \tilde{D}_2$ then $(\tilde{D}_1, \tilde{D}_2) = (D_1, D_2) - \mu_1\mu_2$ where $\mu_i = \mu_p(D_i)$ for $i = 1, 2$.*

Remark 9.21. Note that this intersection pairing reduces to the topological intersection pairing for $\Bbbk = \mathbb{C}$.

Proof. Statement 1) and 2) are trivial if D_1, D_2 and D do not pass through p. Otherwise, use the Moving Lemma that leaves the intersection pairing invariant. Moreover, note that the first three statements imply 4) so it is enough to prove 3). Indeed,

$$(D_1, D_2) \overset{1)}{=} (\pi_p^* D_1, \pi_p^* D_2) \overset{\text{Lemma}}{=} (\tilde{D}_1 + \mu_1 E, \tilde{D}_2 + \mu_2 E) \overset{2)}{=} (\tilde{D}_1, \tilde{D}_2) + \mu_1\mu_2 + \mu_2\mu_1 + \mu_1\mu_2(-1)$$

because $(E, \tilde{D}_2) = (E, \pi_p^* D_2) + \mu_2(E, E) = 0 + \mu_2$. [I am a bit lost about what we use exactly.]

To verify 3), consider the local description of $B_p(X)$ around p. If x, y are generators of $M_p \subseteq \mathcal{O}_{X,p}$ then they extend to regular functions on an open neighborhood U of p. Since $\pi_p^{-1}(U) \subseteq B_p(X)$ is given by the equation $xt_1 = yt_0$ on $U \times \mathbb{P}^1$ where $\mathbb{P}^1$ has coordinate functions t_0, t_1. So $E = V(x)$ on $U \times D_+(t_0)$ and $\pi_p^{-1}(U)$ is given by $y = xv$ for $v = \frac{t_1}{t_0}$. Now, consider the irreducible curve given on U by $V(y)$. The point p has multiplicity 1 on $V(y)$.

Let $C := V(y)$. Take $\pi_p^* C$ which is $\tilde{C} + E$ by the Lemma as p has multiplicity 1 on C. So we get

$$0 \overset{2)}{=} (\pi_p^* C, E) = (\tilde{C}, E) + (E, E) = 1 + (E, E)$$

as $E = V(X)$ and $\tilde{C} = V(y)$. □

Theorem 9.22. *Let X be a smooth projective surface and $C \subseteq X$ an irreducible curve. Then there exist finitely many blowups $X_m \to X_{m-1} \to \cdots \to X_1 \to X$ such that the (iterated) strict transform of C on X_m is smooth.*

Remark 9.23. This is not necessarily the same as the normalization, as it is not necessarily a finite morphism. However, this is already explicit and algorithmic.

The theorem has multiple proofs but all are based on some (natural number valued) invariant that measures how singular a curve is and we decrease it in each step.

Proof. We may consider each singular point of C separately. Suppose that p is singular on C. (Note that it is still smooth on the surface) Then take the blowup $\pi_p : B_p(X) \to X$ and the blowup of the curve $\tilde{\pi}_p^* C$ which is $\tilde{C} + \mu_p(C)E$ by the Lemma 9.16. Then

$$0 \overset{2)}{=} (\pi_p^* C, E) = (\tilde{C}, E) + \mu_p(C)(E, E)$$

hence, $(\tilde{C}, E) = \mu_p(C)$. However, we can calculate $(\tilde{C}, E)$ in another way:

$$(\tilde{C}, E) \overset{\text{def}}{=} \sum_{q \in E} i_q(\tilde{C}, E) \geq \sum_{q \in E} \mu_q(\tilde{C})$$

because in an appropriate local coordinate system $E = V(x)$, and if x is part of a system of generators of M_q then $i_q(\tilde{C}, E) = \mu_q(\tilde{C})$ (just expand the definitions). However, it may happen that x is not a part of a system of generators when we only have $i_q(\tilde{C}, E) \geq \mu_q(\tilde{C})$ since in this case x is a sum of powers of generators.

So if $|\tilde{C} \cap E| > 1$ then for $Q \in \tilde{C} \cap E$ we must have $\mu_q < \mu_p$ since we just got $\mu_p(C) - \mu_p(\tilde{C}) \geq \sum_{q \neq p} \mu_q(\tilde{C})$. However, we are not done since it may also happen that $|\tilde{C} \cap E| = 1$ after each blowup. In that case, consider $\overline{\mathcal{O}}_{C,p}$ the integral closure of $\mathcal{O}_{C,p}$ in $K(C)$. Then we claim that $\overline{\mathcal{O}}_{C,p}$ is a finitely generated $\mathcal{O}_{C,p}$-module. Indeed, $\mathcal{O}_{C,p}$ is a localization of some finitely generated $\Bbbk$-algebra A but integral closure commutes with localization hence $\overline{\mathcal{O}}_{C,p}$ is the localization of the integral closure B of A. Since B is a finitely generated A-module, we get that by localization that $\overline{\mathcal{O}}_{C,p}$ is also a finitely generated $\mathcal{O}_{C,p}$-module.

Taking the common denominators of the generators, we get an $f \in \mathcal{O}_{C,p}$ such that $f\overline{\mathcal{O}}_{C,p} \subseteq \mathcal{O}_{C,p}$. Therefore,

$$\dim_{\Bbbk} \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p} \leq \dim_{\Bbbk} \overline{\mathcal{O}}_{C,p}/f\overline{\mathcal{O}}_{C,p}$$

But $\overline{\mathcal{O}}_{C,p}/f\overline{\mathcal{O}}_{C,p}$ is a finitely generated $\mathcal{O}_{C,p}/(f)$ module and $\dim_{\Bbbk} \mathcal{O}_{C,p}/(f) < \infty$ as we have proved this in Lemma 8.8. Therefore, we got $\dim_{\Bbbk} \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p} < \infty$.

We shall show that: First, if q is the unique point of $\tilde{C}$ above P then $\mathcal{O}_{\tilde{C},q} \subseteq \overline{\mathcal{O}}_{C,p}$, and secondly, if q is singular then $\dim \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{\tilde{C},p} < \dim_{\Bbbk} \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p}$. This is enough since then after finitely many steps we must get that q is smooth as the dimension $\dim_{\Bbbk} \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p}$ can decrease only finitely many times.

To prove the first, take $C^{\nu} \to C$ the normalization of C. Note that $C^{\nu} = \tilde{C}^{\nu}$ since they are normal projective curves that are birational. Moreover, we can describe the integral closures as $\overline{\mathcal{O}}_{C,p} = \bigcap_{p^{\nu} \to p} \mathcal{O}_{C^{\nu}, p^{\nu}}$ and similarly for q hence we get $\mathcal{O}_{\tilde{C},q} \subseteq \overline{\mathcal{O}}_{\tilde{C},q} = \overline{\mathcal{O}}_{C,p}$ as they have the same normalization. To prove the second claim, suppose that we have $\dim \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{\tilde{C},p} = \dim_{\Bbbk} \overline{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p}$. Then, since $\mathcal{O}_{C,p} \subseteq \mathcal{O}_{\tilde{C},q}$ we get $\mathcal{O}_{C,p} = \mathcal{O}_{\tilde{C},q}$ as a subring of $K(C)$. However, this is impossible: If $(x, y) = M_p \subseteq \mathcal{O}_{X,p}$ then by the local description o the blowup $xt_1 = yt_0$ on $U \times \mathbb{P}^1$ we get that the maximal ideal $M_q \subseteq \mathcal{O}_{B_p(X),q}$ is generated by x and $\frac{y}{x} = \frac{t_1}{t_0}$. It means that $\frac{y}{x} \in \mathcal{O}_{\tilde{C},q} = \mathcal{O}_{C,p}$. Hence, $y = \frac{y}{x} \cdot x \in \mathcal{O}_{C,p}$ but that is a contradiction as $\mathcal{O}_{C,p}$ has a principal maximal ideal so p must be singular. □

Twelfth Lecture, 1st of December

Theorem 9.24. (Zariski) *Let $\varphi : X \dashrightarrow Y$ be a birational map of smooth projective surfaces. Then there exists a factorization*

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array}$$

where both f and g are compositions of blowup maps, hence Z is also a smooth projective surface.

Remark 9.25.

1. Weak factorization theorem: (Wlodarczyk, early 2000's) Given a map of smooth projective varieties $X \dashrightarrow Y$ over a field $\Bbbk$ of characteristic zero. Then $\varphi = \varphi_1 \circ \ldots \varphi_r$ where φ_i: blowup or blowdown (the rational inverse of a blowup map which is a birational equivalence).

2. A smooth projective surface X is called a (relative) minimal model if for all $X \to Y$ morphism that is also a birational equivalence with Y being a smooth projective surface is an isomorphism. It can be shown that a smooth projective surface becomes a minimal model after finitely many blowdown. Sometimes a blowdown is also called contraction since it contracts a curve $E \subseteq X$ to a point (indeed, this is the exceptional divisor of the blowup). However, this E cannot be arbitrary, e.g. it is forced to be isomorphic to $\mathbb{P}^1$ and it must have $(E, E) = -1$. It is a Theorem of Castelnuovo that the converse is also true: if $E \subseteq X$ such that $E \cong \mathbb{P}^1$and $(E, E) = -1$ then there exists a Y such that $X \cong B_p(Y)$

for some $p \in Y$ and $E = \varphi^{-1}(p)$. Fact: A minimal model is unique for its birational equivalence class, except for surfaces birational to $\mathbb{P}^1 \times$ curve.

Lemma 9.26. *Let X be a normal quasi-projective surface and $\varphi : X \dashrightarrow \mathbb{P}^n$ a rational maps. Then φ is defined outside a finite set of points.*

Proof. For some $U \subseteq X$ open, φ is a morphism $U \to \mathbb{P}^n$. Then $X \backslash U$ is a proper closed subset, i.e. it is a union of curves and points. Let C be an irreducible curve in $X \backslash U$. We show that φ is defined around each $p \in C$. So we may assume that X is affine. Then C corresponds to a prime ideal P_C in $\mathcal{A}_X$. Then $\mathcal{O}_{X,C} = \left(\mathcal{A}_X\right)_{P_C}$ is a discrete valuation ring (i.e. it is normal as X is smooth hence $\mathcal{A}_X$ is integrally closed, of dimension one and Noetherian). Then φ induces a map $K(\mathbb{P}^n) \hookrightarrow K(X)$. As in the curve case, one can prove that there exists an i such that $\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_0} \in \mathcal{O}_{X,C}$. Since $\mathcal{O}_{X,p}$ is a localization of $\mathcal{O}_{X,C}$ for each $p \in C$, we also have $\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_0} \in \mathcal{O}_{X,p}$. Hence, in a neighborhood V of p they define a morphism $V \to D_+(x_i) \subseteq \mathbb{P}^n$ which coincides with φ on $U \cap V$ which is dense open. Consequently, $X \backslash U$ can be chosen to just a union of curves as for every curve, we can extend φ to an open subset of C. ☐

Lemma 9.27. *Let $\varphi : X \to Y$ be a morphism of quasi-projective surfaces that is a birational equivalence and assume that Y is smooth. Assume, moreover, that φ^{-1} is not defined at some $q \in \varphi(X)$. Then the* $\dim \varphi^{-1}(q) = 1$ *i.e. it contains an (irreducible) curve.*

Proof. We may assume that X is affine. Consider the rational map $Y \overset{\varphi^{-1}}{\dashrightarrow} X \hookrightarrow \mathbb{A}^n$. It is given by rational functions $(g_1, \ldots, g_n) \in K(Y)^n$ such that one of them, say $g_1 \notin \mathcal{O}_{Y,q}$. Then write $g_1 = \frac{u}{v}$ for some $u, v \in \mathcal{O}_{Y,q}$ where $v \in M_q$. As Y is smooth, $\mathcal{O}_{Y,q}$ is a unique factorization domain (see Remark 7.4) hence we may assume that u and v have no common factors. Let $C = V(\varphi^* v) \subseteq X$. Besides, $\varphi^* u = x_1 \varphi^* v$ as $x_1 = \varphi^* g_1$. Therefore, $V(\varphi^* u) \supseteq C$. So we got $C \subseteq \varphi^{-1}(Z)$ where $Z = V(u, v) \subseteq Y$. Since u and v have no common factor, Z is a finite set, containing q. [I still don't see why $u(q) = 0$.] Then, by shrinking Y, we may assume that $Z = \{q\}$. ☐

Theorem 9.28. *Let $\varphi : X \to Y$ be a morphism of smooth projective surfaces such that it is a birational equivalence. If $q \in Y$ is a point such that φ^{-1} is not defined at q then there exists a unique factorization*

$$\begin{array}{ccc} X & \overset{\exists! \tilde{\varphi}}{\dashrightarrow} & B_q(Y) \\ & \searrow^{\varphi} & \downarrow \pi \\ & & Y \end{array}$$

This is called the universal property of the blowup.

Remark 9.29. Note φ must be surjective by the assumptions, as Y is projective and the image contains an open dense set.

Corollary 9.30. *If $\varphi : X \to Y$ is as above then there exists a sequence of blowdown $X \to X_n \to \cdots \to X_0 =$ Y such that their composition is φ.*

Proof. The previous lemmas imply that φ^{-1} is not defined at at most finitely many points where the preimages have dimension one. Then φ contracts finitely many curves in X. We prove by induction on the number of these curves. For zero curve, we are done by Lemma 9.27. Otherwise, apply Theorem 9.28 giving a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\varphi}} & B_q(Y) \\ & \searrow^{\varphi} & \downarrow \pi \\ & & Y \end{array}$$

Here, $\tilde{\varphi}$ contracts less curves than φ because π contracts 1 curve. Therefore, we may apply the induction hypothesis on $\tilde{\varphi}$. $\square$

Lemma 9.31. *Assume that $\varphi : X \dashrightarrow Y$ is a birational map of projective surfaces with Y being smooth. If $q \in \varphi(X)$ such that φ^{-1} is not defined at q then there exists a curve $C \subseteq X$ such that $\varphi(C) = q$.*

Proof. Let Z be the closure in $X \times Y$ of the graph of φ. Then we have a diagram

$$\begin{array}{ccc} & Z & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ X & \xrightarrow{\ \varphi\ } & Y \end{array}$$

where π_1 and π_2 are morphisms that are birational equivalences. If φ^{-1} is not defined at q then π_2^{-1} is not defined at q since then we could obtain the inverse of φ at q by composing with π_1. Then, by Lemma 9.27 there exists an irreducible curve $C' \subseteq Z$ such that $\pi_2(C') = q$. Moreover, $C := \pi_1(C')$ is a curve in X as if it were not then C' would be a subset of a product of two one-point sets in $Z \subseteq X \times Y$. $\square$

Remark 9.32. Note that Z is not necessarily smooth in the above lemma, but that is not needed to apply Lemma 9.27.

Proof of Theorem 9.28. Take the blowup $B_q(Y)$ and let $\tilde{\varphi} = \pi^{-1} \circ \varphi : X \dashrightarrow B_q(Y)$ a rational map. We will show that $\tilde{\varphi}$ is a morphism. Assume that $\tilde{\varphi}$ is not defined at $p \in X$. By Lemma 9.27 applied on $\tilde{\varphi}^{-1}$ there exists a curve $C_p \subseteq B_q(Y)$ such that $\tilde{\varphi}^{-1}(q) = \varphi^{-1}\big(\pi(C_p)\big) = \{p\}$. Hence, $\pi(C_p) = \{\varphi(p)\}$ is a point. This can only happen if $\pi(C_p) = q$ and $C_p = E$.

Let $(x, y) = M_q \subseteq \mathcal{O}_{X,q}$. By the local description of $B_q(Y)$, there exists a neighborhood V of q such that $\pi^{-1}(V) = \{xt_1 = yt_0\} \subseteq V \times \mathbb{P}^1$. Hence, $\varphi^{-1}(q)$ contains a curve through p as φ^{-1} is not defined at q so we can apply Lemma 9.31.

We claim that we may choose x, y so that $\varphi^* x \in M_p^2$ where M_p is the maximal ideal of $\mathcal{O}_{X,p}$. Indeed, assume that $\varphi^* x, \varphi^* y \notin M_p^2$. Then $\varphi^* y$ is a local equation for C in $\mathcal{O}_{X,p}$ because $V(\varphi^* y)$ contains only 1 component passing through p. (If not then $V(\varphi^* y) \supseteq V(v) \cup V(w)$ hence $v, w \mid \varphi^* y$ in $\mathcal{O}_{X,p}$ which is a unique factorization domain as X is smooth, so $\varphi^* y \in M_p^2$, which is a contradiction.) $V(\varphi^* x) \supseteq C$ there exists an $f \in \mathcal{O}_{X,p}$ such that $\varphi^* x = f \cdot \varphi^* y$. Then replace x by $x_0 = x - f(p)y$ so $\varphi^* x_0 = \big(f - f(p)\big) \cdot \varphi^*(y) \in M_p^2$. This is again, a contradiction so we got $\varphi^* x \in M_p^2$.

For all $q_0 \in E$ where $\tilde{\varphi}^{-1}$ is defined, we have

$$\left(\tilde{\varphi}^{-1}\right)^* \varphi^* x = \pi^* x = x \in M_{q_0}^2$$

which is a contradiction as x is a local equation. This proves the first part of the theorem. $\square$

Remark 9.33. If we know the resolution of singularities of surfaces, then Theorem 9.28 implies Theorem 9.24. Indeed, let Z be the closure of the graph of p in $X \times Y$. Then $\tilde{Z} \to Z$ is a resolution of singularities as we can compose the projection $\tilde{Z} \to Z$ by the projections of Z to X and Y and apply Theorem 9.28.

Theorem 9.34. *If X is a smooth projective surface and $\varphi : X \dashrightarrow \mathbb{P}^n$ is a rational map then there exists a factorization $X_m \overset{\pi_m}{\to} X_{m-1} \overset{\pi_{m-1}}{\to} X_{m-2} \to \cdots \to X_1 \overset{\pi_0}{\to} X_0 = X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ such that their composition is a morphism.*

Proof. We consider the linear system of hyperplanes $|H|$ on $\mathbb{P}^n$ and we pull it back to a linear system $\varphi^*|H|$ on X. We have seen that φ is a morphism if and only if $\varphi^*|H|$ is base point free, i.e. $\cap_{D \in \varphi^*|H|} \mathrm{Supp}(D) = \emptyset$. We will show that even if φ is not a morphism, we can blow it up in a way that $\varphi^*|H|$ gets closer to being base point free.

Let $\mathrm{d}(\varphi) = (D, D)$ for $D \in \varphi^*|H|$. Note that it is indeed an invariant of φ as the divisors in $\varphi^*|H|$ just differ by a divisor of the form $\mathrm{div}(f)$. Moreover, $\mathrm{d}(\varphi) \geq 0$ as, by the Moving Lemma 9.16 we can assume that they have no common component so then the intersection multiplicity form is computed by i_p's which are nonnegative.

If $D \in \varphi^*|H|$ and $p \in \mathrm{Supp}(D)$ then consider the maps

$$\begin{array}{ccc} B_p(X) & \xrightarrow{\pi} & X \\ & \overset{\tilde{\varphi}}{\searrow} & \downarrow \varphi \\ & & \mathbb{P}^n \end{array}$$

applying the universality of the blow up, Theorem 9.28 and taking inverses. We claim that $\mathrm{d}(\tilde{\varphi}) < \mathrm{d}(\varphi)$ if $\mathrm{d}(\varphi) > 0$. This is enough since then after finitely many blowups, we achieve $\mathrm{d}(\tilde{\varphi}) = 0$ hence $\cap_{D \in \tilde{\varphi}^*|H|} \mathrm{Supp}(D) = \emptyset$ meaning that the resulting map is already a morphism.

By the lemma we have $\pi^* D = \tilde{D} + \mu_p(D) E$ hence

$$\mathrm{d}(\tilde{\varphi}) = (\tilde{D}, \tilde{D}) = (\pi^* D, \pi^* D) + \big(\mu_p(D) E, \mu_p(D) E\big) = \mathrm{d}(\varphi) - \mu_p(D)^2$$

where $\mu_p(D) \geq 1$ so we got the claim. $\square$