Prefix-reversal Gray codes

Alexey Medvedev

Sobolev Institute of Mathematics, Novosibirsk, Russia

joint work with

Elena Konstantinova, Sobolev Institute of Mathematics

Symmetries of Graphs and Networks IV

Rogla, Slovenia, June 29–July 5, 2014

Gray codes

Combinatorial Gray codes [J. Joichi et al., (1980)]

A combinatorial Gray code is now referred as a method of generating combinatorial objects so that successive objects differ in some pre-specified, usually small, way.

[D.E. Knuth, The Art of Computer Programming, Vol.4 (2010)]

Knuth recently surveyed combinatorial generation:

Gray codes are related to

efficient algorithms for exhaustively generating combinatorial objects.

(tuples, permutations, combinations, partitions, trees)

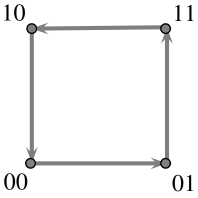
Examples

Hamming cube H_n [F. Gray, (1953), U.S. Patent 2,632,058]

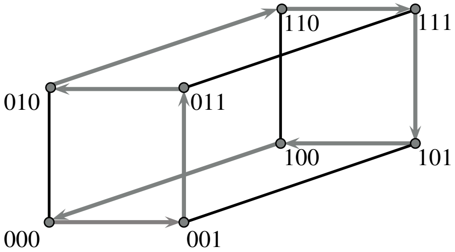
The first Gray code was introduced relative to binary strings

$n = 2$: 00 01 | 11 10

$n = 3$: 000 001 011 010 | 110 111 101 100



H_2



H_3

Examples

Symmetric group Sym_n [R. Eggleton, W. Wallis, (1985); D. Rall, P. Slater, (1987)]

The group of permutations:

Q: Is it possible to list all permutations in a list so that each one differs from its predecessor in every position?

A: YES!

[1234]	[3124]	[2314]
[4123]	[4312]	[4231]
[2341]	[1243]	[3142]
[3412]	[2431]	[1423]
[1324]	[3214]	[2134]
[4132]	[4321]	[4213]
[3241]	[2143]	[1342]
[2413]	[1432]	[3421]

Generating permutations in Sym_4

Gray codes: generating permutations

[S. Zaks, (1984)]

Zaks' algorithm:

each successive permutation is generated by reversing a suffix of the preceding permutation.

Describe in terms of prefixes:

- *Start with* $I_n = [12\dots n]$*;*
- *Let* ζ_n *be the sequence of sizes of these prefixes defined by recursively as follows:*

$$\zeta_2 = 2$$
$$\zeta_n = (\zeta_{n-1}\, n)^{n-1}\, \zeta_{n-1},\ n > 2,$$

  where a sequence is written as a concatenation of its elements;
- *Flip prefixes according to the sequence.*

Zaks' algorithm: examples

If $n = 2$ then $\zeta_2 = 2$ and we have:

$$[\underline{12}] \quad [21]$$

If $n = 3$ then $\zeta_3 = 23232$ and we have:

$$[\underline{12}3] \quad [\underline{31}2] \quad [\underline{23}1]$$

$$[\underline{213}] \quad [\underline{132}] \quad [321]$$

If $n = 4$ then $\zeta_4 = 23232423232423232423232$ and we have:

$$[\underline{12}34] \quad [\underline{41}23] \quad [\underline{34}12] \quad [\underline{23}41]$$

$$[\underline{213}4] \quad [\underline{142}3] \quad [\underline{431}2] \quad [\underline{324}1]$$

$$[\underline{31}24] \quad [\underline{24}13] \quad [\underline{13}42] \quad [\underline{42}31]$$

$$[\underline{132}4] \quad [\underline{421}3] \quad [\underline{314}2] \quad [\underline{243}1]$$

$$[\underline{23}14] \quad [\underline{12}43] \quad [\underline{41}32] \quad [\underline{34}21]$$

$$[\underline{3214}] \quad [\underline{2143}] \quad [\underline{1432}] \quad [4321]$$

Greedy Gray code: generating permutations

[A. Williams, J. Sawada, (2013)]

Describe in terms of prefixes:

- *Start with* $I_n = [12\dots n]$*;*
- *Take the largest size prefix we can flip not repeating a created permutation;*
- *Flip this prefix.*

Example: for $n = 4$ then we have

$$[\overline{1234}]\ [\overline{432}1]\ [\overline{2341}]\ [\overline{143}2]\ [\overline{3412}]\ [\overline{214}3]\ [\overline{4123}]\ [\overline{32}14]$$

$$[\overline{2314}]\ [\overline{413}2]\ [\overline{3142}]\ [\overline{241}3]\ [\overline{1423}]\ [\overline{324}1]\ [\overline{4231}]\ [\overline{13}24]$$

$$[\overline{3124}]\ [\overline{421}3]\ [\overline{1243}]\ [\overline{342}1]\ [\overline{2431}]\ [\overline{134}2]\ [\overline{4312}]\ [\overline{21}34]$$

Prefix–reversal Gray codes: generating permutations

Each 'flip' is formally known as prefix–reversal.

The Pancake graph P_n

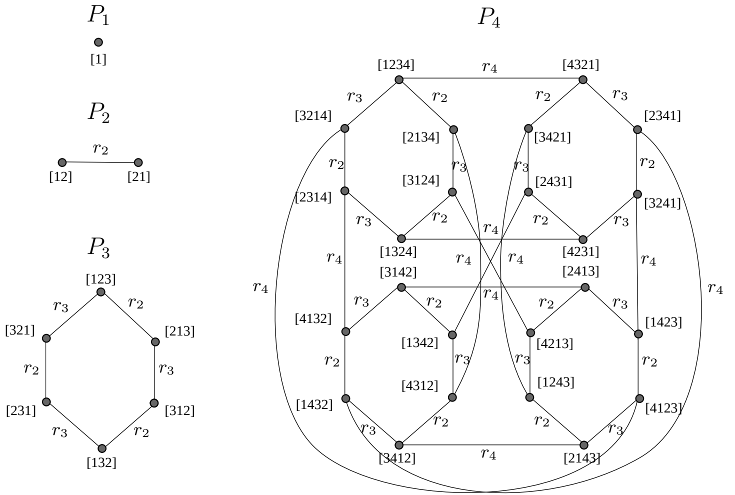
is the Cayley graph on the symmetric group Sym_n with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\pi_1 \dots \pi_i \pi_{i+1} \dots \pi_n] r_i = [\pi_i \dots \pi_1 \pi_{i+1} \dots \pi_n]$.

Cycles in P_n [A. Kanevsky, C. Feng, (1995); J.J. Sheu, J.J.M. Tan, K.T. Chu, (2006)]

All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.

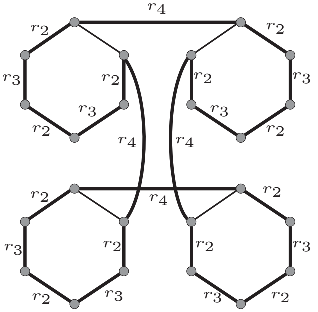
Pancake graphs: hierarchical structure

P_n consists of n copies of $P_{n-1}(i) = (V^i, E^i)$, $1 \leqslant i \leqslant n$, where the vertex set V^i is presented by permutations with the fixed last element.



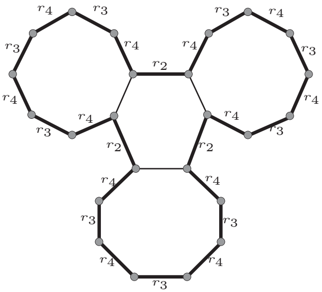
Both algorithms are based on independent cycles in P_n.

Zaks' prefix–reversal Gray code:
$(r_2\, r_3)^3$ – flip the minimum number of topmost pancakes that gives a new stack.



(a) Zaks' code in P_4

Williams' prefix–reversal Gray code:
$(r_n\, r_{n-1})^n$ – flip the maximum number of topmost pancakes that gives a new stack.



(b) Williams' code in P_4

Independent cycles in P_n

Theorem 1. (K., M.)

The Pancake graph $P_n, n \geqslant 4$, contains the maximal set of $\frac{n!}{\ell}$ independent ℓ–cycles of the canonical form

$$C_\ell = (r_n\, r_m)^k, \tag{1}$$

where $\ell = 2\,k$, $2 \leqslant m \leqslant n-1$ and

$$k = \begin{cases} O(1) & \textit{if} \quad m \leqslant \lfloor \frac{n}{2} \rfloor; \\ O(n) & \textit{if} \quad m > \lfloor \frac{n}{2} \rfloor \quad \textit{and} \quad n \equiv 0 \pmod{n-m}; \\ O(n^2) & \textit{else.} \end{cases} \tag{2}$$

Corollary

The cycles presented in Theorem 1 have no chords.

Hamilton cycles based on small independent even cycles

Hamilton cycle or path in P_n $\Rightarrow$ PRGC

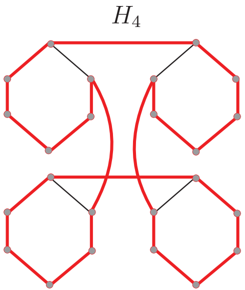
Definition

The Hamilton cycle H_n **based on independent ℓ–cycles** *is called a Hamilton cycle in P_n, consisting of paths of lengths $l = \ell - 1$ of independent cycles, connected together with external to these cycles edges.*

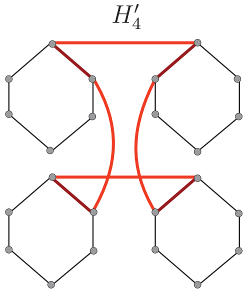
Hamilton cycles based on small independent even cycles

Definition

The complementary cycle H'_n to the Hamilton cycle H_n based on independent cycles is defined on unused edges of H_n and the same external edges.



(c) Hamilton cycle H_4 in P_4



(d) Complementary cycle H'_4 in P_4

Non-existence of Hamilton cycles

Suppose the complementary cycle H'_n has form $(r_m\, r_j)^t$, where $m \in \{2, \dots, n\}$, $r_j \in PR \backslash \{r_m\}$.

Theorem 2. (K., M.)

The only Hamilton cycles H_n based on independent cycles from Theorem 1 with the complementary cycle H'_n of form $(r_m\, r_j)^t$, where $m \in \{2, \dots, n\}$, are Zaks', Greedy and Hamilton cycle based on $(r_4\, r_2)^4$ in P_4.

Proof. $H'_n = (r_m\, r_j)^t \Rightarrow H'_n$ has form from Theorem 1. Thus, the following inequality should hold

$$2\frac{n!}{L_{\max}} \leqslant L_{\max}, \qquad (3)$$

where $L_{\max}$ is the maximal length of cycles from Theorem 1.

Non-existence of Hamilton cycles

The length L_{max} can be estimated as

$$L_{\text{max}} \leqslant n(n+2),$$

and therefore

$$2n! \leqslant L_{\text{max}}^2,$$

$$n! \leqslant \frac{1}{2}n^2(n+2)^2.$$

The inequality does not hold starting from $n = 7$. For n from 4 to 6 it is easy to verify using the exact lengths that inequality holds only for $n = 4$.
□

Non-existence of Hamilton cycles

Suppose the complementary cycle H'_n has form $H'_n = (r_m r_\xi)^t$, where by r_ξ we mean that every second reversal may be different from previous. Another way of thinking of it is to treat r_ξ as a random variable taking values in $PR \backslash \{r_n, r_m\}$ with some distribution.

Theorem 3. (K., M.)

The only Hamilton cycles H_n based on independent cycles from Theorem 1 with the complementary cycle H'_n of form $(r_m r_\xi)^t$, where $m \neq \{n, n-2\}$ and $r_\xi \in$ PR$\backslash\{r_n, r_m\}$ is Greedy Hamilton cycle in P_n.

Proof is based on structural properties of the graph, hierarchical structure and length's argument above.

Remark. Existence in the case $m = n - 2$ is only unresolved when $\ell = O(n)$.

Hamilton cycles based on small independent even cycles

Open problem

Suppose the complementary cycle H'_n *has form* $H'_n = (r_\eta \, r_\xi)^t$, *where* $r_\eta \in \{r_n, r_m\}$ *and* $r_\xi \in PR \backslash \{r_n, r_m\}$.

PRGC: hierarchical construction

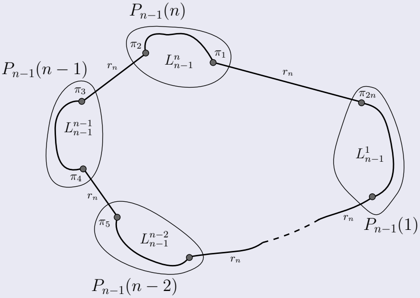
Hierarchical construction

Suppose we know a bunch of Hamilton cycle constructions in graph P_{n-1}. Then the PRGC can be constructed using the complementary $2n$–path passing through all copies of P_{n-1} in P_n exactly once.

Example:

1) Zaks' construction:

$$H_n'^1 = (r_n \, r_{n-1})^n$$



PRGC based on large independent cycles

Theorem 4. (K., M.)

There are no Hamilton cycles in P_n, $n \geqslant 4$, *based on independent* $\frac{n!}{2}$*–cycles but there are Hamilton paths based on the following two independent cycles:*

$$C_n^1 = ((C_{n-1}^1/r_{n-1})r_n)^n,$$
$$C_n^2 = ((C_{n-1}^2/r_{n-1})r_n)^n,$$

where $C_4^1 = (r_3\, r_2\, r_4\, r_2\, r_3\, r_4)^2$ *and* $C_4^2 = (r_2\, r_3\, r_4\, r_3\, r_2\, r_4)^2$.

Proof is based on the hierarchical structure of P_n and on the non-existence 4–cycles in P_n.

Thank you for your attention!