1 Introduction

An important and perhaps interesting topic in nonlinear analysis and convex optimization concerns solving inclusions of the form $0 \in A(x)$, where $A$ is a maximal monotone operator on a Hilbert space $H$. Its importance in convex optimization is evidenced from the fact that many problems that involve convexity can be formulated as finding zeros of maximal monotone operators. For example, convex minimizations and convex-concave mini-max problems, to mention but a few, can be formulated in this way. In particular, the subdifferential of a proper, convex and lower semi-continuous (lsc) function $f$, $\partial f$, is a maximal monotone operator and a point $p \in H$ minimizes $f$ if and only if $0 \in \partial f(p)$. One of the most powerful and versatile solution techniques for solving variational inequalities, convex minimizations, and convex-concave mini-max (saddle-point) problems is the proximal point algorithm (PPA).

The PPA was first introduced by B. Martinet (1970) and it is based on the notion of the proximal mapping $J_\beta(x) = x_\beta = \text{arg min}\{f(z) + \|z - x\|^2/2\beta : z \in H\}$, introduced by J. J. Moreau (1965). For the problem of minimizing a proper, lower semi-continuous convex function $f$ on a Hilbert space, the proximal point algorithm in exact form generates a sequence $\{x_n\}$ by taking the $(n+1)$th iterate to be the minimizer of $f(x) + \|x - x_n\|^2/2\beta_n$, where $\beta_n > 0$. It was shown by Y. Censor and S. A. Zenios (1992) that the quadratic additive term appearing above can be replaced by more general D-functions which resembles (but are not strictly) distance functions. They characterized the properties of such D-functions which when used in the proximal minimization algorithm preserve its convergence. It was further shown by J. Eckstein (1993) that for every Bregman function (a strictly convex differentiable function that induces the distance measure or a D-function on the Euclidean space) there exists a “nonlinear” version of the PPA.

Many mathematicians have studied the PPA, and other iterative processes such as the Mann and the Mann-Ishikawa iteration processes for solving nonlinear operator equations. They investigated the convergence of such iterative processes and in some cases gave the rate of convergence of such methods. Among them, the work of R. T. Rockafellar (1976), O. Güler (1991), C. D. Ha (1990), P. Tseng (2000), H. K. Xu (2002) and P. Tossings (1994), is worth mentioning. Other methods for finding zeros of operators have been shown to be strongly connected with the above mentioned methods. For instance, J. Eckstein and D. P. Bertsekas (1992) showed by means of an operator called a “splitting operator” that the Douglas-Rachford splitting method for finding a zero of the sum of two operators is a special case of the PPA. They observed that applications of Douglas-Rachford splitting, such as the alternating direction method of multipliers for convex
programming decomposition, are also special cases of the PPA, an observation which allows the unification and generalization of a variety of convex programming algorithms.

2 Proximal Point Algorithms

For a maximal monotone operator $A$, consider the following set valued equation

$$0 \in A(x).$$

As pointed out earlier, one method for finding the zeros of (1) is the PPA, which starts at an arbitrary point $x_0 \in H$ and generates recursively a sequence of points

$$x_{n+1} = (I + \beta_n A)^{-1}(x_n) + e_n, \quad \text{for all } n \geq 0,$$

where $\{\beta_n\} \subset (0, \infty)$ and $\{e_n\}$ is considered to be the error sequence. Güler [7] constructed an example showing that Rockafellar’s algorithm 2 with $e_n = 0$ for all $n \geq 0$ does not converge strongly, in general. Since weak convergence is not enough for an efficient algorithm and the PPA does not converge strongly in general, much of research have been devoted to finding algorithms which will always converge strongly, or at least modify Rockafellar’s algorithm in such a way that strong convergence is guaranteed. One such modification have been obtained by Solodov and Svaiter [15].

In an attempt to obtain strong convergence, Solodov and Svaiter proposed an algorithm which generates a sequence $\{x_n\}$ satisfying

$$x_{n+1} = P_{H_n \cap W_n}x_0, \quad \text{for all } n \geq 0,$$

where $H_n \cap W_n$ is the projection of $H$ onto $H_n \cap W_n$ where $H_n := \{z \in H : \langle z - y_n, v_n \rangle \leq 0\}$ and $W_n := \{z \in H : \langle z - x_n, x_0 - x_n \rangle \leq 0\}$.

It was proved in [15] that if the sequence $\{\mu_n\}$ is bounded from above, then the sequence $\{x_n\}$ constructed above converges strongly to $P_{A^{-1}(0)}x_0$. Though their algorithm is strongly convergent, it needs more computing time since it requires at each iterate, to calculate a projection, a task which may not always be easy. Xu’s idea was to construct a less time consuming algorithm which still converge strongly. In view of Halpern’s algorithm, Xu [20] proposed the following algorithm

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n) + e_n, \quad \text{for all } n \geq 0,$$

and showed that algorithm 3 converge strongly provided that $\{e_n\} \in \ell^1$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ of real numbers are chosen appropriately. In [2], we showed that strong convergence is still ensured even if $x_0$ is replaced by any arbitrary point $u$ of $H$ (not necessarily the starting point of the PPA) and $\{e_n\} \in \ell^p$ for $1 \leq p < 2$. 

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Recently, Takahashi [16] studied the PPA in a Banach space by the viscosity approximation method, where the \((n+1)\)th iterate was given as

\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(I + \beta_n A)^{-1}(x_n),
\]

where \(f : C \to C\) is a strict contraction (a-contraction with \(0 < a < 1\)) defined on a nonempty closed convex subset \(C\) of a reflexive Banach space. It is clear that in a Hilbert space setting, we can generalize the result under algorithm 3 by the viscosity approximation method even when one takes into account the error terms in \(\ell^1\), see [2]. Exploring the case when \(f\) is nonexpansive leads to several interesting results, some of which guarantees strong convergence of the modified PPA. We have discussed this case in [2].

3 Further Investigations

First we will consider Xu’s modified algorithm (algorithm 1 of [2]) and hope to prove a strong convergence result under general errors. More precisely, we will show that for \(\|e_n\| \to 0\) and \(\beta_n \to \infty\) the sequence generated by algorithm 1 is strongly convergent. This will lead us to the following open question: can one design a PPA by choosing appropriate regularization parameters \(\alpha_n\) such that strong convergence of \(\{x_n\}\) is preserved, for \(\|e_n\| \to 0\) and \(\beta_n\) bounded? We will also try to investigate and give the convergence rates of some algorithms.

Secondly, we note that Marino et al. [10] proved some convergence results of the Mann iterative process for strict pseudo-contractions without any error terms. However, since errors are bound to occur in any practical algorithm, we will investigate the results of [10] taking into account the errors. Xu [19] proposed a regularization method for the PPA which essentially includes the prox-Tikhonov method of Lehdili and Moudafi [9]. The algorithm proposed in [19] is easily seen to be equivalent to algorithm 1 of [2]. On the other hand, following the ideas contained in [2], one can generalize Xu’s regularization method by considering the case when \(I\) is replaced by any nonexpansive map \(f\), and in this case the resulting algorithm can not be reduced to any of the algorithms of section 3 of [2] (except of course when \(f = I\)).

Lastly, we intend to consider iterative processes for semigroups, and prove some convergence theorems associated with them.

References


