Liquidity Risk and the Dynamics of Arbitrage Capital: Extensions and Robustness

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Abstract

In this supplementary note to our paper, Kondor and Vayanos (2014), we extend our baseline model by allowing for positive supply of assets, stochastic hedgers’ demand and infinitely lived CARA hedgers and show that the main results of our analysis do not change.
First, we extend our model to positive supply. A second extension is to allow the supply \( u \) coming from hedgers to be stochastic. A third extension is to assume that hedgers derive utility from intertemporal consumption rather than from instantaneous changes in wealth. We take their utility to be of the constant absolute risk aversion (CARA) type. In each part, we present both analytical and numerical results.

1 Positive supply

First, we show how the expressions characterizing the equilibrium are influenced by positive supply. Then, we illustrate that our main findings on liquidity and liquidity risk are robust to this variation.

1.1 Equilibrium expressions with positive supply

It is easy to see that in the short-term insurance contract set-up, positive supply changes very little in the derivation. Indeed, in that case only the sum of \( (s + u) \) matters, not its composition. However, we cannot use our trick to move between the short-horizon contracts and long-horizon contracts. To see this, observe that we cannot find \( S_t, X_t, Y_t \) solving (5.6) and

\[
\begin{align*}
\sigma x_t &= (\sigma s_t + \sigma) X_t \\
\sigma y_t &= (\sigma s_t + \sigma) Y_t
\end{align*}
\]

because if we could then the following equations would hold by market clearing

\[
\sigma s = \sigma y_t + \sigma x_t = (\sigma s_t + \sigma) X_t + (\sigma s_t + \sigma) Y_t = (\sigma s_t + \sigma) s
\]

which would imply that \( \sigma s_t = 0 \). However, this is inconsistent with a non-constant wealth process and market clearing.

Thus, we have to derive the equilibrium from first principles. We summarize the characterization of the equilibrium in the next proposition.

**Proposition 1.1** If

\[
V(\tilde{w}_t, w_t) = q(w_t) \tilde{w}_t^{1-\gamma} \frac{1}{1-\gamma}
\]
is the value function for $\gamma \neq 1$, and

$$V(\tilde{w}_t, w_t) = \frac{1}{\rho} \log(\tilde{w}_t) + q_1(w_t)$$ (1.2)

for $\gamma = 1$, where $q(w_t)$ and $q_1(w_t)$ are scalar functions of $w_t$ and

$$S(w_t) = \frac{D - \alpha \Sigma(u + s)}{r} + g(w_t) \Sigma(u + s),$$ (1.3)

then the functions $q(w_t), q_1(w_t)$ and $g(w_t)$ must solve the system of ODEs

$$\rho q = \gamma q^{1 - \frac{1}{\gamma}} + \left(r - q^{1 - \frac{1}{\gamma}}\right) q' w + rq(1 - \gamma) +$$

$$+ \frac{1}{2} \left(q'' + \frac{2q' \gamma}{w} - \frac{2q'^2}{q} + \frac{q(1 - \gamma)\gamma}{w^2}\right) \frac{\alpha^2}{\left(A(w_t) + \alpha - \alpha g'(w) (u + s)^\top \Sigma^\top s\right)^2} (u + s)^\top \Sigma (u + s)$$ (1.4)

or, for $\gamma = 1$,

$$\rho q_1 = \log(\rho) + \frac{r}{\rho} + (r - \rho) q_1 + \frac{1}{2} \left(q'' + \frac{2q' q_1}{w} + \frac{1}{\rho w^2}\right) \frac{\alpha^2}{\left(\alpha + \frac{1}{w} - \alpha g'(w) (u + s)^\top \Sigma^\top s\right)^2} (u + s)^\top \Sigma (u + s)$$ (1.5)

and

$$\frac{\alpha A(w_t)}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s} =$$

$$= \left(r - q(w_t)^{-\frac{1}{\gamma}}\right) w_t g'(w) + \frac{\alpha^2}{\alpha + A(w_t)} \frac{(u + s)^\top \Sigma (u + s)}{\left(\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s\right)^2} \frac{1}{2} g''(w_t) + \alpha - rg(w_t)$$

where we set $q(w_t)^{-\frac{1}{\gamma}} = \rho$ for $\gamma = 1$ and the definition of $A(w_t)$ and the boundary conditions do not change. Furthermore, in equilibrium

$$E_t(dR_t) = \frac{A(w_t) \alpha}{A(w_t) + \alpha - \alpha g'(w) (u + s)^\top \Sigma^\top s} \left(f(w_t) (u + s)^\top \Sigma (u + s) + 1\right) \Sigma (u + s)$$ (1.6)
\[ c_t = q(w_t)^{-\frac{1}{\gamma}} w_t \]

\[ (\sigma_S + \sigma) X_t + \sigma u = \frac{A(w_t)}{A(w_t) + \alpha} b_t \]

\[ (\sigma_S + \sigma) Y_t = \frac{\alpha}{A(w_t) + r\alpha} b_t \]

where

\[ b_t = \frac{(A(w_t) + \alpha) \sigma (u + s)}{A(w_t) + \alpha - \alpha g'(w) (u + s)^\top \Sigma^\top s} \]

and

\[ \sigma_{S_t} = \frac{\alpha g'(w_t) (u + s)^\top \Sigma^\top s}{A(w_t) + \alpha - \alpha g'(w) (u + s)^\top \Sigma^\top s} \sigma (u + s) (u + s)^\top \Sigma^\top. \]

and

\[ f(w_t) = \frac{\alpha g'(w_t)}{A(w_t) + \alpha - \alpha g'(w) (u + s)^\top \Sigma^\top s} \]

**Corollary 1.1** When \( \gamma = 0 \), (1.4) is

\[ \rho - r = -rA(w_t) w_t - \frac{1}{2} \frac{(A'(w_t) + A^2(w_t)) \alpha^2}{A(w_t) + \alpha - \alpha g'(w) (u + s)^\top \Sigma^\top s} (u + s)^\top \Sigma (u + s) \]

### 1.2 Liquidity and liquidity risk with positive supply

The next proposition shows the expression for illiquidity \( \lambda_{nt} \).

**Proposition 1.2** Illiquidity \( \lambda_{nt} \) is equal to

\[ \left( 1 + \frac{A(w_t)}{\alpha} + g'(w)(u + s)^\top \Sigma u \right) \left( \frac{\alpha}{p} - g(w_t) \right) \Sigma_{nn}. \]
Note from (1.6) that the expected return of asset \( n \) is proportional to the covariance \((\Sigma (u + s))_n\) between the asset and portfolio \( u \). We show that this is exactly proportional to the beta of the asset’s return with respect to aggregate illiquidity. Therefore, a liquidity factor based on return covariance with respect to aggregate illiquidity would explain expected returns perfectly in our model. Indeed, the covariance between asset returns and aggregate illiquidity \( \Lambda_t \equiv \sum_{n=1}^{N} \lambda_{nt} \) is

\[
\frac{\text{Cov}_t(d\Lambda_t, dR_t)}{dt} = C^\Lambda(w_t)\Sigma (u + s),
\]

where \( C^\Lambda(w_t) \) is a negative coefficient. Given that both the expected returns in (1.6) and this covariance are proportional to \( \Sigma (u + s) \), we can decompose the expected return of any asset \( n \) into a premium for this liquidity factor, \( \Pi^\Lambda(w_t) \) and the liquidity beta as

\[
\frac{E_t(dR_t)}{dt} = \Pi^\Lambda(w_t)\frac{\text{Cov}_t(d\Lambda_t, dR_t)}{dt}.
\]

On Figures 1-2 we compare the \( s = 0 \) to the \( s = 0.5 \) case in a parametrization where the total volatility in the \( s = 0 \) case, \((u^\top \Sigma u)^{1/2}\), is the same as the total volatility in the \( s = 0.5 \) case \(((u + s)^\top \Sigma (u + s))^{1/2}\). That is, both are kept at 15%. This implies that with no arbitrageurs, the two economy would coincide. Figure 1 is the \( \gamma = 0 \) case, while Figure 2 is the \( \gamma = 1 \) case. It is apparent that the qualitative features of the equilibrium do not change. The main effect of introducing positive supply is that now arbitrageurs have to hold more risk ‘overnight’ as hedgers live only for an instant. This is why the Sharpe-ratio, the expected return, return volatility and liquidity premia all go up.

2 Stochastic demand from hedgers

In this part, we modify the baseline model by allowing for stochastic demand of hedgers. First, we show that it is possible to introduce stochastic \( u \) and yet preserve much of the tractability of our model provided that we restrict the variance \( u_t^\top \Sigma u_t \) of hedgers’ endowment to remain constant over time.

Such specification for \( u_t \) exists. Suppose that \( u \) is stochastic and follows the Ito process

\[
du_t = \mu_u dt + \sigma_u^\top dB_{ut},
\]

where \( \mu_u \) and \( \sigma_u \) are constants. This allows for a simple decomposition of the return into a drift and a diffusion component, which is useful for analytical and numerical analysis.
where $B_{ut}$ is a $K$-dimensional Brownian motion, possibly independent of $B_t$. It is easy to check from Ito’s lemma, that the conditions for a constant $u_t^\top \Sigma u_t$ are
\begin{align}
u_t^\top \Sigma \mu_{ut} + \frac{1}{2} \text{Tr} \left( \sigma_{ut} \Sigma \sigma_{ut}^\top \right) &= 0, \quad \text{(2.15)} \\
u_t^\top \Sigma \sigma_{ut}^\top &= 0. \quad \text{(2.16)}
\end{align}

As a relatively simple example, consider the specification $\mu_{u,t} = -\kappa (u_t - \bar{u})$ and $\sigma_{u,t} = \sqrt{\kappa} \psi_t$, where $\bar{u}$ is a $N \times 1$ vector and $\psi_t$ is a $N \times N$ matrix. We can write (2.15) as
\begin{align}
\psi_t \Sigma u_t &= 0. \quad \text{(2.17)}
\end{align}

One can check that the following $\psi_t$ satisfies (2.17):
\begin{align}
\psi_t = C_t \begin{pmatrix}
\sum_{j \neq 1} (\Sigma u_t)_j & -(\Sigma u_t)_1 & \cdots & -(\Sigma u_t)_1 \\
-(\Sigma u_t)_2 & \sum_{j \neq 2} (\Sigma u_t)_j & \cdots & -(\Sigma u_t)_2 \\
\vdots & \vdots & \ddots & \vdots \\
-(\Sigma u_t)_N & -(\Sigma u_t)_N & \cdots & \sum_{j \neq N} (\Sigma u_t)_j
\end{pmatrix}, \quad \text{(2.18)}
\end{align}

where $C_t$ is
\begin{align}
C_t &= \frac{u_t^\top \Sigma (u_t - \bar{u})}{\sum_{i=1}^N \left( (\sum_{j=1}^N (\Sigma u_t)_j)^2 - 2 (\sum_{j \neq 1} (\Sigma u_t)_j)(\Sigma u_t)_1 + \sum_{j=1}^N \sum_{i,j} (\Sigma u_t)_i \Sigma u_t - (\Sigma u_t)_i \right)}. \quad \text{(2.19)}
\end{align}

In what follows we do not use this particular specification for $\sigma_{u,t}$. We only use the assumption that $u_t$ follows the mean reverting process
\begin{align}
du_t = -\kappa u_t dt + \sigma_{u_t}^\top dB_{ut} \quad \text{(2.20)}
\end{align}

and $\sigma_{u_t}^\top$ is such that $\dot{U} = u_t^\top \Sigma u_t$ is constant. The following proposition characterizes the equilibrium under this assumption.

**Proposition 2.3** If $u$ is stochastic following (2.20) and if (1.1) and (1.2) are the value functions for $\gamma \neq 1$, and $\gamma = 1$ respectively and
\begin{align}
S(w_t) &= \bar{D} - \alpha \Sigma u_t + g(w_t) \Sigma u_t, \quad \text{(2.21)}
\end{align}

then the functions $q(w_t), q_1(w_t)$ and $g(w_t)$ must solve the system of ODEs
\begin{align}
pq = \gamma q^{-\frac{1}{\gamma}} + \left( r - q^{-\frac{1}{\gamma}} \right) q' w + r(1 - \gamma) + \frac{1}{2} \left( q'' + 2q' A(w_t) + \frac{q(1 - \gamma)\gamma}{w^2} \right) \frac{\alpha^2}{(A(w_t) + \alpha)^2} \dot{U}, \quad \text{(2.22)}
\end{align}
or, for $\gamma = 1$,
\[
\rho q_1 = \log(\rho) + \frac{r - \rho}{\rho} + (r - \rho)q'_1 + \frac{1}{2} \left( q''_1 + \frac{2q'_1}{w} \right) \left( \frac{\alpha^2}{(\alpha + \frac{1}{w})^2} \right) \hat{U}
\]
(2.23)

and
\[
g'(w_t)(r - q(w_t) - \frac{1}{\gamma})w_t + \frac{1}{2} g''(w_t) \frac{\alpha^2 \hat{U}}{(\alpha + A(w_t))^2} + \alpha - (r + \kappa)g(w_t) = \frac{\alpha A(w_t)}{\alpha + A(w_t)}
\]
(2.24)

where we set $q(w_t)^{-\frac{1}{\gamma}} = \rho$ for $\gamma = 1$ and the definition of $A(w_t)$ and the boundary conditions do not change. Furthermore, in equilibrium
\[
\frac{E_t(dR_t)}{dt} = \frac{A(w_t) \alpha}{A(w_t) + \alpha} \left( f(w_t) \hat{U} + 1 \right) \Sigma u_t
\]
(2.25)

and
\[
c_t = q(w_t)^{-\frac{1}{\gamma}}w_t
\]

\[
(\sigma_{S_t} + \sigma) X_t + \sigma u = \frac{A(w_t)}{A(w_t) + \alpha} \Sigma u_t
\]
(2.26)

\[
(\sigma_{S_t} + \sigma) Y_t = \frac{\alpha}{A(w_t) + r \alpha} \sigma u_t
\]
(2.27)

and
\[
\sigma_{S_t} = \frac{\alpha g'(w_t)}{A(w_t) + \alpha} \sigma u_t u_t^\top \Sigma^\top.
\]
(2.28)

and
\[
f(w_t) = \frac{\alpha g'(w_t)}{A(w_t) + \alpha}
\]

Note that stochastic $u_t$ does not affect the ODE for $g(w_t)$. Because the expression for $S_t$ has slightly changed, Illiquidity $\lambda_{nt}$ modifies to
\[
\left( 1 + \frac{A(w_t)}{\alpha} + g'(w) \hat{U} \right) \left( \frac{\alpha}{r + \kappa} - g(w_t) \right) \Sigma_{nn}.
\]
(2.29)
Note also that as the expressions for the equilibrium portfolio choices and the price dynamics remains unaffected, expected returns can still be decomposed as 1.13. That is, the liquidity risk factor maintains full explanatory power.

On Figures 3-4 we compare the baseline case with an economy with the same parameters but \( \kappa = 0.01 \). Figure 3 is the \( \gamma = 0 \) case, while Figure 4 is the \( \gamma = 1 \) case. It is apparent that the qualitative features of the equilibrium do not change.

3 Infinitely lived CARA hedgers

First, we show how the expressions characterizing the equilibrium are influenced by infinitely lived hedgers. Then, we illustrate that our main findings on liquidity and liquidity risk are robust to this variation.

3.1 Equilibrium expressions with CARA hedgers

Suppose that, additionally to positive supply and long-lived assets, instead of mean-variance hedgers, hedgers live forever and maximize the utility function

\[
-\mathbb{E}_t \left( \int_t^\infty e^{-\alpha c_{hs}} e^{-\rho_h(s-t)} ds \right)
\]

where \( c_{hs} \) is consumption at \( s \geq t \) and \( \rho_h \) is a subjective discount rate.

The following proposition characterizes the equilibrium.

**Proposition 3.4** If (1.1)-(1.2) are the value functions of arbitrageurs and

\[
V^h(v_t, w_t) = -e^{-[r v_t + F(w_t)]},
\]

is the value function of hedgers where \( F(w_t) \) is a function of arbitrageur wealth, and the price is given by

\[
S(w_t) = \frac{\bar{D}}{r} - \alpha \Sigma (u + s) + g(w_t) \Sigma (u + s),
\]
, then the functions \( q(w_t), q_1(w_t), g(w_t), F(w_t) \) must solve the system of ODEs

\[
\rho q = \gamma q^{1 - \frac{1}{\gamma}} + \left( r - q^{\frac{1}{\gamma}} \right) q' w + rq(1 - \gamma) +
\]

\[
+ \frac{1}{2} \frac{\left( q'' + \frac{2q'\gamma}{w} - \frac{2q'^2}{q} + \frac{q(1-\gamma)\gamma}{w^2} \right) (r\alpha)^2}{\left( A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u \right)^\top \Sigma (u + s)}
\]

or, for \( \gamma = 1 \),

\[
\rho q_1 = \log(q) + \frac{r - p}{\rho} + (r - \rho) q_1' + \frac{1}{2} \frac{\left( q''_1 + \frac{2q'_1}{w} + \frac{r}{\rho w^2} \right) (r\alpha)^2}{\left( \frac{1}{w} - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u \right)^\top \Sigma u (u + s)}
\]

and

\[
A(w_t) \alpha
\]

\[
= \frac{A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u}{A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u}
\]

\[
= \left( r - q(w_t)^{-\frac{1}{\gamma}} \right) w_1 g'(w_t) + \frac{(r\alpha)^2 (u + s)^\top \Sigma (u + s)}{A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u (u + s)} \frac{1}{2} \frac{q''(w_t)}{w_t}
\]

\[
+ r\alpha - r g(w_t).
\]

and

\[
0 = \rho - r - rF(w_t) + r \log(r) + rau \Sigma u^\top \Sigma u + F'(w_t)(r - q(w_t)^{-\frac{1}{\gamma}})w_t
\]

\[
+ \frac{(r\alpha)^2}{2} \left( \frac{F''(w_t)}{A(w_t)} \right) \frac{A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u (u + s)}{A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u (u + s)}
\]

\[
- \frac{A(w_t) (r\alpha)^2}{A(w_t) - F'(w_t) + r\alpha - r\alpha g'(w) (u + s) \Sigma u (u + s)} u \Sigma (u + s).
\]

where we set \( q(w_t)^{-\frac{1}{\gamma}} = \rho \) for \( \gamma = 1 \) and the definition of \( A(w_t) \) and the boundary conditions do not change.
Furthermore, in equilibrium

$$
E_t(dR_t) = \frac{A(w_t) r \alpha}{A(w_t) - F'(w_t) + r \alpha - r \alpha g'(w)(u + s) \Sigma s} \left(f (w_t)(u + s) \Sigma (u + s) + 1\right) \Sigma (u + s)
$$

(3.37)

and

$$
c_t = q(w_t)^{-\frac{1}{\gamma}} w_t
$$

$$
(\sigma_{S_t} + \sigma) X_t + \sigma u = \frac{(A(w_t) - F'(w_t))}{A(w_t) - F'(w_t) + r \alpha} b_t
$$

(3.38)

$$
(\sigma_{S_t} + \sigma) Y_t = \frac{r \alpha}{A(w_t) - F'(w_t) + r \alpha} b_t
$$

(3.39)

where

$$b_t = \frac{(A(w_t) - F'(w_t) + r \alpha) (u + s) \sigma (u + s)}{A(w_t) - F'(w_t) + r \alpha - r \alpha g'(w)(u + s) \Sigma s}.
$$

(3.40)

and

$$\sigma_{S_t} = \frac{r \alpha g'(w_t)}{A(w_t) + r \alpha - F'(w_t) - r \alpha g'(w)(u + s) \Sigma s} (u + s) \Sigma \top
$$

(3.41)

and

$$f(w_t) = \frac{r \alpha g'(w_t)}{A(w_t) - F'(w_t) + r \alpha - r \alpha g'(w)(u + s) \Sigma s}.
$$

(3.42)

3.2 Liquidity and liquidity risk with CARA hedgers

The next proposition shows how $\lambda_{nt}$ changes in this variant.

**Proposition 3.5** Illiquidity $\lambda_{nt}$ is equal to

$$
\left(1 + \frac{A(w_t) - F'(w_t)}{r \alpha} + g'(w)(u + s) \Sigma u\right) (\alpha - g(w_t)) \Sigma_{mn}.
$$

(3.42)
Clearly, the same argument that expected returns can be decomposed as (1.13) still holds.

On Figure 5 we compare the case with short-lived hedgers to the case with long-lived hedgers. In general, we keep the parametrization from the positive supply variant. That is, $s = 0.5$ and

$$\left( (u + s)^\top \Sigma (u + s) \right)^{\frac{1}{2}} = 15\%, \ r = 2\%, \ \rho_h = \rho = 4\%.$$  

While the short-lived hedgers case we keep $\alpha = 2$, we choose $r\alpha = 2$ in the long-lived hedgers economy. Therefore, the two economy coincide without arbitrageurs. For brevity, we show here the $\gamma = 1$ case only. (Note that comparing the short-lived hedgers case on Figure 5 and the $s = 0.5$ case on Figure 1 gives the difference between the $\gamma = 1$ and $\gamma = 0$ cases.)

It is apparent that the qualitative features of the equilibrium do not change.
APPENDIX—Proofs

Proof of Proposition 1.1 Given that the dynamics of hedgers’ and arbitrageurs’ wealth follows an Ito process where the drifts and diffusions are

\[
\mu_{vt} = rv_t - c_{lt} + X_t^T (\mu_{st} + \bar{D} - rS_t) + u^T \bar{D}, \tag{A.1}
\]

\[
\sigma_{vt} = \left( X_t^T \left( \sigma_{st}^T + \sigma^T \right) + u^T \sigma^T \right),
\]

for the hedger and

\[
\mu_{wt} = (rw_t - c_t + Y_t^T (\mu_{st} + \bar{D} - rS_t)), \tag{A.2}
\]

\[
\sigma_{wt} = Y_t^T \left( \sigma_{st}^T + \sigma^T \right),
\]

for arbitrageurs, the first-order condition for the hedger gives

\[
(\sigma_{st} + \sigma) X_t + \sigma u = \frac{A(w_t)}{A(w_t) + \alpha} b_t \tag{A.3}
\]

where

\[
b_t \equiv (\sigma_{st} + \sigma) s + \sigma u.
\]

Turning to arbitrageurs, the Bellman equation is

\[
\rho V = \max_{\hat{c}_t, \hat{Y}_t} \left\{ u(\hat{c}_t) + V_{\hat{w}_t} \hat{c}_t + \frac{1}{2} V_{\hat{w}_t} \sigma_{\hat{w}_t}^T \sigma_{\hat{w}_t} + V_{\bar{w}_t} \mu_{\bar{w}_t} + \frac{1}{2} V_{\bar{w}_w} \sigma_{\bar{w}_w}^T \sigma_{\bar{w}_w} \right\}, \tag{A.4}
\]

where \( u(\hat{c}_t) = \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} \) for \( \gamma \neq 1 \) and \( u(\hat{c}_t) = \log(\hat{c}_t) \) for \( \gamma = 1 \), \((\mu_{\hat{w}_t}, \sigma_{\hat{w}_t})\) are the drift and diffusion of the arbitrageur’s own wealth \( \hat{w}_t \), and \((\mu_{\bar{w}_t}, \sigma_{\bar{w}_t})\) are the drift and diffusion of the arbitrageurs’ total wealth.

When \( \gamma \neq 1 \), we substitute (1.1) and (A.1)-(A.2) into (A.4) to write it as

\[
\rho q(w_t) \tilde{w}_t^{1-\gamma} = \max_{\hat{c}_t, \hat{Y}_t} \left\{ \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} + q(w_t) \tilde{w}_t^{-\gamma} \left( r\tilde{w}_t - \hat{c}_t + \hat{Y}_t^T (\mu_{st} + \bar{D} - rS_t) \right) \right. \\
- \frac{1}{2} q(w_t) \gamma \tilde{w}_t^{-\gamma-1} \hat{Y}_t^T \left( \sigma_{st}^T + \sigma^T \right) (\sigma_{st} + \sigma) \hat{Y}_t \\
+ q'(w_t) \tilde{w}_t^{-\gamma} (r\tilde{w}_t - c_t + Y_t^T (\mu_{st} + \bar{D} - rS_t)) \\
+ \frac{1}{2} q''(w_t) \tilde{w}_t^{-\gamma} Y_t^T \left( \sigma_{st}^T + \sigma^T \right) (\sigma_{st} + \sigma) Y_t \\
\left. + q''(w_t) \tilde{w}_t^{-\gamma} Y_t^T \left( \sigma_{st}^T + \sigma^T \right) (\sigma_{st} + \sigma) Y_t \right\}
\]
The first-order conditions with respect to $\hat{c}_t$ and $\hat{Y}_t$ are

$$\hat{c}_t = q(w_t)^{-\frac{1}{\gamma}} \hat{w}_t$$

(A.6)

$$\hat{Y}_t = \frac{\hat{w}_t}{\gamma} \left( \left( \sigma_{S_t}^\top + \sigma^\top \right) (\sigma_{S_t} + \sigma) \right)^{-1} (\mu_{S_t} + \bar{D} - rS_t) + \frac{q'(w_t)}{q(w_t)} Y_t$$

respectively. When $\gamma = 1$, we substitute (1.2),(A.1)-(A.2) into (A.4) to write it as

$$\rho \left( \frac{1}{\rho} \log(\hat{w}_t) + q_1(w_t) \right) = \max_{\dot{c}_t, \dot{Y}_t} \left\{ \log(\dot{c}_t) + \frac{1}{2\rho w_t^2} \dot{Y}_t^\top (\sigma_{S_t}^\top + \sigma^\top) (\sigma_{S_t} + \sigma) \dot{Y}_t + q'(w_t) (r\dot{w}_t - \dot{c}_t + \dot{Y}_t^\top (\mu_{S_t} + \bar{D} - rS_t)) + \frac{1}{2} q''(w_t) Y_t^\top (\sigma_{S_t}^\top + \sigma^\top) (\sigma_{S_t} + \sigma) Y_t \right\}$$

The first-order conditions with respect to $\dot{c}_t$ and $\dot{Y}_t$ are (A.6) for $q(w_t) = \frac{1}{\rho}$. Since in equilibrium $\dot{c}_t = c_t$ and $\dot{w}_t = w_t$, (A.6) implies that

$$c_t = q(w_t)^{-\frac{1}{\gamma}} w_t,$$

(A.7)

and using the definition of $A(w_t) = \frac{\gamma}{w_t - \frac{q'(w_t)}{q(w_t)}}$, we find

$$\left( \sigma_{S_t}^\top + \sigma^\top \right) Y_t = \frac{\left( (\sigma_{S_t} + \sigma) \right)^{-1} (\mu_{S_t} + \bar{D} - rS_t)}{A(w_t)}.$$

(A.8)

Then, (A.3), (A.8) and the market clearing equation gives

$$\left( \sigma_{S_t}^\top + \sigma^\top \right)^{-1} (\mu_{S_t} + \bar{D} - rS_t) = \frac{A(w_t) \alpha}{A(w_t) + \alpha} b_t$$

(A.9)

implying also

$$Y_t^\top (\mu_{S_t} + \bar{D} - rS_t) = \frac{A(w_t) \alpha^2}{(A(w_t) + \alpha)^2} b_t^\top b_t$$

(A.10)

$$X_t^\top (\mu_{S_t} + \bar{D} - rS_t) = \left( \frac{A(w_t) \alpha (A(w_t))}{(A(w_t) + \alpha)^2} b_t^\top b_t \right) - \frac{A(w_t) \alpha}{A(w_t) + \alpha} u^\top \sigma^\top b_t$$

(A.11)
Substituting (A.9)-(A.11) and (A.7) and the definition of \( A(w_t) \) into (A.5) and simplifying gives

\[
\rho q = \gamma q^{1-\frac{1}{\gamma}} + \left( r - q^{1-\frac{1}{\gamma}} \right) q' w + r q (1 - \gamma) + \frac{1}{2} \left( q'' + \frac{2q' \gamma}{w} - \frac{2q^2}{q} + \frac{q(1 - \gamma) \gamma}{w^2} \right) \frac{\alpha^2}{(A(w_t) + \alpha)^2} b_t^\top b_t.
\]

Before, we simplify (A.12) further, we derive the ODE for \( g(w_t) \):

Note that from Ito’s Lemma

\[
\begin{align*}
\mu_{St} &= \mu_{wt} S'(w_t) + \frac{1}{2} \sigma_{wt}^2 S''(w_t) = (r w_t - q(w_t)^{-\frac{1}{\gamma}} w_t + \frac{A(w_t) \alpha^2}{(A(w_t) + \alpha)^2} b_t^\top b_t) S'(w_t) \\
&+ \frac{1}{2} (r \alpha)^2 - \frac{1}{2} \frac{\alpha^2}{(A(w_t) + \alpha)^2} b_t^\top b_t S''(w_t) = \\
&= \left( r - q(w_t)^{-\frac{1}{\gamma}} \right) w_t S'(w_t) + \frac{\alpha^2}{(A(w_t) + \alpha)^2} b_t^\top b_t \left( A(w_t) S'(w_t) + \frac{1}{2} S''(w_t) \right),
\end{align*}
\]

and

\[
\begin{align*}
\sigma_{St} &= \sigma_{wt} S'(w_t) = (\sigma_{Si} + \sigma) Y t S'(w_t) \\
&= \frac{\alpha}{A(w_t) + \alpha} b_t S'(w_t)^\top, \tag{A.13}
\end{align*}
\]

Substituting back \( \sigma_{Si} \) into the definition of \( b_t \), we get

\[
\begin{align*}
b_t &= \frac{r \alpha}{A(w_t) F'(w_t) + r \alpha} b_t S'(w_t)^\top s + \sigma (u + s) \\
b_t &= \frac{\sigma (u + s)}{\left( 1 - \frac{r \alpha}{A(w_t) - F'(w_t) + r \alpha} S'(w_t)^\top s \right)}.
\end{align*} \tag{A.15}
\]

and using (3.32), we get (1.9) and (1.10). Substituting (1.9) and (1.10) into (A.9), we find the ODE for \( g(w_t) \) (1.4). Also, given (1.9), and (A.12) simplifies to (1.4). Using the same steps for the \( \gamma = 1 \) case, we get (1.5). Finally, using (1.9) and (1.10) and the definition of \( \frac{E_t(dR_t)}{dt} \) and (A.9), we get (1.6).
Proof of Proposition 1.2: From (3.32)

\[ \frac{\partial S_{nt}}{\partial u_n} = - \left( \frac{\alpha}{r} - g(w_t) \right) \Sigma_{nn}. \]

For \( \frac{\partial X_{nt}}{\partial u_n} \), we use (A.8),(A.9),(A.14) and (1.9) to rewrite

\[ (\sigma_S + \sigma) Y_t = \frac{\alpha}{\alpha + A(w_t)} b_t \]  

as

\[ \sigma \left( \frac{1}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s \alpha} g'(w) (u + s) (u + s)^\top \Sigma^\top + I \right) Y_t = \]

\[ = \frac{\alpha}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s \alpha} \sigma (u + s) \]

multiplying both sides by \( \sigma^{-1} \) shows that \( Y_t \) must be collinear with \( (u + s) \). So let us write \( Y_t = (u + s) v \) giving

\[ \frac{\sigma \alpha}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s \alpha} g'(w) (u + s) \Sigma^\top (u + s) v + (u + s) v = \]

\[ = \frac{\alpha}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s \alpha} \sigma (u + s) \]

\[ \frac{\alpha g'(w) (u + s)^\top \Sigma^\top (u + s) v}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s \alpha} + v = \frac{\alpha}{\alpha + A(w_t) - g'(w_t) (u + s)^\top \Sigma^\top s \alpha} \]

which implies

\[ v = \frac{\alpha}{\alpha g'(w) (u + s)^\top \Sigma^\top u + \alpha + A(w_t)}. \]  

(A.17)

Hence, by market clearing

\[ \frac{\partial X_{nt}}{\partial u_n} - \frac{\partial Y_{nt}}{\partial u_n} = \frac{1}{g'(w) (u + s)^\top \Sigma^\top u + 1 + \frac{A(w_t)}{\alpha}} \]

giving the result.
Proof of Proposition 2.3  We show only the proof for the \( \gamma \neq 1 \) case, the \( \gamma = 1 \) case comes analogously by using the corresponding arguments for the constant \( u \) variant.

Suppose that the price follows the Ito process

\[
dS_t = \mu_S dt + \sigma_S^T dB_t + \sigma_{Su}^T dB_{ut}. \tag{A.18}
\]

The budget constraint of hedgers can be written as

\[
dv_t = (rv_t - c_{ht}) dt + X_t^T (\mu_S + \bar{D} - rS_t) dt + \left( X_t^T (\sigma_S + \sigma)^T + u_t^T \sigma^T \right) dB_t + X_t^T \sigma_{Su}^T dB_{ut}
\]

the FOC of hedgers is

\[
\mu_t + \bar{D} - rS_t - \alpha \left[ (\sigma_S + \sigma)^T ([\sigma_S + \sigma] X_t + \sigma u_t) + \sigma_{Su}^T \sigma_{Su} X_t \right] = 0. \tag{A.19}
\]

Now turning to arbitrageurs, their wealth follows

\[
dw_t = rw_t dt + Y_t^T (\mu_S + \bar{D} - rS_t) dt + Y_t^T (\sigma_S + \sigma)^T dB_t + Y_t^T \sigma_{Su}^T dB_{ut} - c_t dt
\]

and their Bellman equation is given by

\[
\rho V = \max_{\hat{c}_t, \hat{Y}_t} \left\{ u(\hat{c}_t) + V_{\hat{w}_t}(\hat{\mu}_t, \hat{\sigma}_t) + \frac{1}{2} V_{\hat{w}_t}(\hat{\sigma}_u t) \sigma_{\hat{w}_t} + \hat{V}_{\hat{w}_t}(\mu_{\hat{u}} t + \sigma_{\hat{u}} t) \right\}, \tag{A.20}
\]

where \( u(\hat{c}_t) = \frac{\hat{c}_t^{\gamma-1}}{1-\gamma} \) for \( \gamma \neq 1 \), \( (\hat{\mu}_t, \hat{\sigma}_t) \) are the drift and diffusion of the arbitrageur’s own wealth \( \hat{w}_t \), and \( (\mu_{\hat{u}}, \sigma_{\hat{u}} t) \) are the drift and diffusion of the arbitrageurs’ total wealth.

Using \( V(\hat{w}_t, \hat{w}_t) = q(w_t)^{\hat{w}_t^{\gamma-1} \hat{w}_t}, (A.20) \) may be written as

\[
\rho q(w_t)^{\hat{w}_t^{\gamma-1} \hat{w}_t} = \max_{\hat{c}_t, \hat{Y}_t} \left\{ \frac{\hat{w}_t^{\gamma}}{1-\gamma} + q(w_t) \hat{w}_t^{\gamma} \left( r\hat{w}_t - \hat{c}_t + \hat{Y}_t^T (\mu_S + \bar{D} - rS_t) \right) \right\}
\]

\[
+ \frac{1}{2} q'(w_t) \hat{w}_t^{\gamma-1} \left[ \hat{Y}_t^T \left( \sigma_{\hat{w}_t} + \sigma \right) \left( \sigma_{\hat{w}_t} + \sigma \right) \hat{Y}_t + \hat{Y}_t^T \sigma_{\hat{u}} \sigma_{\hat{u}} \hat{Y}_t \right]
\]

\[
+ q'(w_t) \hat{w}_t^{\gamma-1} \left[ \hat{Y}_t^T \left( \sigma_{\hat{w}_t} + \sigma \right) \left( \sigma_{\hat{w}_t} + \sigma \right) \hat{Y}_t + \hat{Y}_t^T \sigma_{\hat{u}} \sigma_{\hat{u}} \hat{Y}_t \right]
\]

The first-order conditions with respect to \( \hat{c}_t \) and \( \hat{Y}_t \) are

\[
\hat{c}_t = q(w_t)^{-\frac{1}{\gamma}} \hat{w}_t \tag{A.22}
\]

\[
\hat{Y}_t = \frac{\hat{w}_t}{\gamma} \left( \left( \sigma_{\hat{w}_t} + \sigma \right) \left( \sigma_{\hat{w}_t} + \sigma \right) + \sigma_{\hat{u}} \sigma_{\hat{u}} \right)^{-1} (\mu_S + \bar{D} - rS_t) + q'(w_t) \hat{w}_t \tag{A.23}
\]

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respectively.

Now turning to the price process, from (2.21) and (A.18) using Ito’s Lemma that

\[ \mu_{S_t} = g'(w_t) \Sigma u_t + \frac{1}{2} \left[ \sigma_{w_t} \sigma_{w_t} + \sigma_{w_{u_t}} \sigma_{w_{u_t}} \right] g''(w_t) \Sigma u_t - \kappa \left( g(w_t) - \frac{\alpha}{r + \kappa} \right) \Sigma u_t \]

\[ \sigma_{S_t}^T = g'(w_t) \sigma_{w_t} \Sigma u_t \quad (A.24) \]

\[ \sigma_{S_{u_t}}^T = g'(w_t) \sigma_{w_{u_t}} \Sigma u_t + \left( g(w_t) - \frac{\alpha}{r + \kappa} \right) \Sigma \sigma_{u_t} \quad (A.25) \]

Combining the diffusion terms in the process for arbitrageurs’ wealth with (A.24) and (A.25) the diffusion terms can be expressed as

\[ \sigma_{w_t}^T = \frac{Y_t^T \sigma}{1 - g'(w_t) Y_t^T \Sigma u_t} \quad (A.26) \]

\[ \sigma_{w_{u_t}}^T = \frac{\left( g(w_t) - \frac{\alpha}{r + \kappa} \right) Y_t^T \Sigma \sigma_{u_t}}{1 - g'(w_t) Y_t^T \Sigma u_t} \quad (A.27) \]

Using the market clearing condition, \( X_t + Y_t = 0 \) together with (A.19) and (A.23) one finds

\[ \mu_{S_t} + D - rS_t = \frac{A(w_t) \alpha}{A(w_t) + \alpha} \left( \sigma_{S_t}^T \sigma u_t + \Sigma u_t \right) = \frac{A(w_t) \alpha}{A(w_t) + \alpha} \frac{\Sigma u_t}{1 - g'(w_t) Y_t^T \Sigma u_t}. \quad (A.28) \]

Now using (A.23) again we find

\[ \frac{A(w_t) \alpha}{A(w_t) + \alpha} \frac{\Sigma u_t}{1 - g'(w_t) Y_t^T \Sigma u_t} = A(w_t) \left[ (\sigma_{S_t} + \sigma)^T (\sigma_{S_t} + \sigma) Y_t + \sigma_{S_{u_t}}^T \sigma_{S_{u_t}} Y_t \right] \quad (A.29) \]

Substituting out \( \sigma_{S_t} \) and \( \sigma_{S_{u_t}} \) from (A.29) using (A.24)-(A.27) with some algebra one arrives to
\[
\frac{\alpha}{\alpha + A(w_t)} \Sigma u_t = \frac{g'(w_t) Y_t^\top \Sigma Y_t}{1 - g'(w_t) Y_t^\top \Sigma u_t} \Sigma u_t + \Sigma Y_t + \sigma_{S_t}^\top \sigma_{u_t} \Sigma Y_t
\]  
\hspace{1cm} \text{(A.30)}

Premultiplying both sides of (A.30) by \(\sigma_{u_t}\) and using \(\sigma_{u_t} \Sigma u_t = 0\) one finds that \(Y_t\) is collinear with \(u_t\). Assuming \(Y_t = \nu_t u_t\) we find

\[
\nu_t = \frac{\alpha}{\alpha + A(w_t) + \alpha g'(w_t) U}.
\]

Then substituting \(Y_t = \nu_t u_t\) into (A.28) gives (2.25) also implying that

\[
Y_t^\top (\mu_{S_t} + D - rS_t) = \frac{A(w_t) \alpha^2}{(A(w_t) + \alpha)^2} \tilde{U}.
\]  
\hspace{1cm} \text{(A.31)}

Then, substituting (2.25) and, using market clearing, the right hand side of (A.29) into (A.19), we get (2.28). The latter, together with \(Y_t = \nu_t u_t\), also implies (2.26) and (2.27). Substituting \(Y_t = \nu_t u_t\) into (A.26) and (A.27) and then simplifying (2.25) gives (2.24).

Turning to \(q(w_t)\), substituting (2.27) and (A.31) into (A.21) and simplifying gives (2.22).

**Proof of Proposition 3.4** The Bellman equation of a hedger is

\[
\rho_h V^h = \max_{c_{ht}, z_{ht}} \left\{ u(c_{ht}) + V^h_{V^h} + \frac{1}{2} V^h_{V^h} \sigma_{c_{ht}}^\top \sigma_{c_{ht}} + V^h_{\mu_{c_{ht}}} + \frac{1}{2} V^h_{\mu_{c_{ht}}} \sigma_{c_{ht}}^\top \sigma_{c_{ht}} \right\},
\]

\hspace{1cm} \text{(A.32)}

where \(u(c_{ht}) = e^{-\alpha c_{ht}}\), \((\mu_{c_{ht}}, \sigma_{c_{ht}})\) are the drift and diffusion of the hedger’s wealth \(v_t\), and \((\mu_{uc}, \sigma_{uc})\) are the drift and diffusion of arbitrageur wealth given by (A.1) and (A.2).

We substitute (3.31) and (A.1)-(A.2) into (A.32) to write it as

\[
-\rho e^{-[r a v + F(w_t)]} = \max_{c_{ht}, z_{ht}} \left\{ \begin{array}{c}
-\alpha c_{ht} + r a e^{-[r a v + F(w_t)]} (r V_t - c_{ht} + X_t^\top (\mu_{S_t} + D - rS_t) + u_t^\top D) \\
-\frac{1}{2} (r \alpha)^2 e^{-[r a v + F(w_t)]} (X_t^\top (\sigma_{S_t}^\top + \sigma^\top) + u_t^\top \sigma^\top) (X_t^\top (\sigma_{S_t}^\top + \sigma^\top) + u_t^\top \sigma^\top)^\top \\
+ F'(w_t) e^{-[r a v + F(w_t)]} (r V_t - c_t + Y_t^\top (\mu_{S_t} + D - rS_t)) \\
+ \frac{1}{2} [F''(w_t) - F'(w_t)^2] e^{-[r a v + F(w_t)]} (Y_t^\top (\sigma_{S_t}^\top + \sigma^\top) (\sigma_{S_t}^\top + \sigma^\top)^\top Y_t \\
- r \alpha F'(w_t) e^{-[r a v + F(w_t)]} (X_t^\top (\sigma_{S_t}^\top + \sigma^\top) + u_t^\top \sigma^\top) (\sigma_{S_t}^\top + \sigma^\top)^\top Y_t \\
\end{array} \right\}
\]

\hspace{1cm} \text{(A.33)}
respectively. Which we can rewrite as

$$e^{-\alpha c_{hs}} = r e^{\left[-\rho \alpha w_t + F(w_t)\right]},$$

(A.34)

$$\mu_{S_t} + \bar{D} - r_{S_t} = r \alpha \left(\sigma_{S_t}^\top + \sigma^\top\right) \left(X_t^\top \left(\sigma_{S_t}^\top + \sigma^\top\right) + u^\top \sigma^\top\right)^\top + F'(w_t) \left(\sigma_{S_t}^\top + \sigma^\top\right) \left(\sigma_{S_t}^\top + \sigma^\top\right)^\top Y_t,$$

(A.35)

respectively. Which we can rewrite as

$$c_{hs} = rv_t + \frac{F(w_t)}{\alpha} - \frac{\log(r)}{\alpha},$$

(A.36)

$$(\sigma_{S_t} + \sigma) X_t + \sigma u = \frac{\left(\sigma_{S_t}^\top + \sigma^\top\right)^{-1} \left(\mu_{S_t} + \bar{D} - r_{S_t}\right)}{r \alpha} - \frac{F'(w_t) \left(\sigma_{S_t}^\top + \sigma^\top\right)^\top Y_t}{r \alpha}$$

(A.37)

respectively. Using (A.36) and (A.37), we can simplify (A.33) to

$$0 = \rho - r - r F(w_t) + r \log(r) + r \alpha X_t^\top \left(\mu_{S_t} + \bar{D} - r_{S_t}\right) + r \alpha u^\top \bar{D}$$

$$- \frac{1}{2} (r \alpha)^2 \left(\left(X_t^\top \left(\sigma_{S_t}^\top + \sigma^\top\right) + u^\top \sigma^\top\right) \left(\left(\sigma_{S_t} + \sigma\right) X_t + \sigma u\right) + F'(w_t) \left(r_{w_t} - c_t + Y_t^\top \left(\mu_{S_t} + \bar{D} - r_{S_t}\right)\right) + F''(w_t) \left(Y_t^\top \left(\sigma_{S_t}^\top + \sigma^\top\right)\right) \left(\sigma_{S_t} + \sigma^\top\right)^\top Y_t\right)

- \frac{1}{2} \left[F''(w_t) - F'(w_t)^2\right] \left(Y_t^\top \left(\sigma_{S_t}^\top + \sigma^\top\right)\right) \left(\sigma_{S_t} + \sigma^\top\right)^\top Y_t$$

Turning to arbitrageurs, the Bellman equation is

$$\rho V = \max_{\delta_t, \bar{u}_t} \left\{ u(\delta_t) + V_{\hat{w}_t} \mu_{\hat{w}_t} + \frac{1}{2} V_{\hat{w}_t} \hat{w}_{\hat{w}_t}^\top \sigma_{\hat{w}_t} \hat{w}_{\hat{w}_t} + V_{w_t} \mu_{w_t} + \frac{1}{2} V_{w_t} \sigma_{w_t} \sigma_{w_t} + V_{\hat{w}_t} \sigma_{\hat{w}_t} \sigma_{\hat{w}_t} \right\},$$

(A.38)

where $u(\delta_t) = \frac{\delta_t^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ and $u(\delta_t) = \log(\delta_t)$ for $\gamma = 1$, $(\mu_{\hat{w}_t}, \sigma_{\hat{w}_t})$ are the drift and diffusion of the arbitrageur’s own wealth $\hat{w}_t$, and $(\mu_{\hat{w}_t}, \sigma_{\hat{w}_t})$ are the drift and diffusion of the arbitrageurs’ total wealth.

When $\gamma \neq 1$, we substitute (1.1) and (A.1)-(A.2) into (A.38) to write it as

$$\rho q(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} = \max_{\delta_t, \bar{u}_t} \left\{ \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} + q(w_t) \hat{w}_t^{-\gamma} \left(r_{\hat{w}_t} - c_t + \hat{Y}_t^\top \left(\mu_{\hat{w}_t} + \bar{D} - r_{\hat{w}_t}\right)\right)\right.$$

$$- \frac{1}{2} q(w_t) \gamma \hat{w}_t^{-\gamma-1} \hat{Y}_t^\top \left(\sigma_{\hat{w}_t}^\top + \sigma^\top\right) \left(\sigma_{\hat{w}_t} + \sigma\right) \hat{Y}_t + q'(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} \left(r_{\hat{w}_t} - c_t + \hat{Y}_t^\top \left(\mu_{\hat{w}_t} + \bar{D} - r_{\hat{w}_t}\right)\right)$$

$$+ \frac{1}{2} q''(w_t) \frac{\hat{w}_t^{1-\gamma}}{1-\gamma} \hat{Y}_t^\top \left(\sigma_{\hat{w}_t}^\top + \sigma^\top\right) \left(\sigma_{\hat{w}_t} + \sigma\right) \hat{Y}_t + q'(w_t) \hat{w}_t^{-\gamma} \hat{Y}_t^\top \left(\sigma_{\hat{w}_t}^\top + \sigma^\top\right) \left(\sigma_{\hat{w}_t} + \sigma\right) \hat{Y}_t$$

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The first-order conditions with respect to $\hat{c}_t$ and $\hat{Y}_t$ are

$$\hat{c}_t = q(w_t)^{-\frac{1}{\gamma}} \hat{w}_t$$  \hspace{1cm} (A.40)

$$\hat{Y}_t = \frac{\hat{w}_t}{\gamma} \left( \left( \sigma_{S_t}^T + \sigma^T \right) (\sigma_{S_t} + \sigma) \right)^{-1} (\mu_{S_t} + D - rS_t) + \frac{q'(w_t)}{q(w_t)} Y_t$$

, respectively.

When $\gamma = 1$, we substitute (1.2), (A.1)-(A.2) into (A.38) to write it as

$$\rho \left( \frac{1}{\rho} \log(\hat{w}_t) + q_1(w_t) \right) = \max_{\hat{c}_t, \hat{Y}_t} \left\{ \frac{1}{\rho} \log(\hat{c}_t) + \frac{1}{\rho \hat{w}_t} \left( r \hat{w}_t - \hat{c}_t + \hat{Y}_t^T (\mu_{S_t} + D - rS_t) \right) - \frac{1}{2 \rho \hat{w}_t} \hat{Y}_t^T \left( \sigma_{S_t}^T + \sigma^T \right) (\sigma_{S_t} + \sigma) \hat{Y}_t + q'_1(w_t) (rw_t - c_t + Y_t^T (\mu_{S_t} + D - rS_t)) + \frac{1}{2} q''_1(w_t) Y_t^T \left( \sigma_{S_t}^T + \sigma^T \right) (\sigma_{S_t} + \sigma) Y_t \right\}$$  \hspace{1cm} (A.41)

The first-order conditions with respect to $\hat{c}_t$ and $\hat{Y}_t$ are (A.40) for $q(w_t) = \frac{1}{\rho}$. Since in equilibrium $\hat{c}_t = c_t$ and $\hat{w}_t = w_t$, (A.40) implies that

$$c_t = q(w_t)^{-\frac{1}{\gamma}} w_t,$$  \hspace{1cm} (A.42)

and using the definition of $A(w_t) = \frac{\gamma}{w_t} - \frac{q'(w_t)}{q(w_t)}$, we find

$$\left( \sigma_{S_t}^T + \sigma^T \right) Y_t = \frac{\left( (\sigma_{S_t} + \sigma) \right)^{-1} (\mu_{S_t} + D - rS_t)}{A(w_t)}. $$  \hspace{1cm} (A.43)

Then, (A.37) and (A.43) and the market clearing equation

$$\frac{\left( \sigma_{S_t}^T + \sigma^T \right)^{-1} (\mu_{S_t} + D - rS_t)}{r \alpha} - \frac{F'(w_t)}{r \alpha} \left( \sigma_{S_t}^T + \sigma^T \right) Y_t - \sigma u + \frac{\left( (\sigma_{S_t} + \sigma) \right)^{-1} (\mu_{S_t} + D - rS_t)}{A(w_t)} = (\sigma_{S_t} + \sigma) s,$$  \hspace{1cm} (A.44)

form a linear system of three equations in three unknowns of $\left( \sigma_{S_t}^T + \sigma^T \right) (\mu_{S_t} + D - rS_t)$, $(\sigma_{S_t} + \sigma) X_t + \sigma u$, $(\sigma_{S_t}^T + \sigma^T) Y_t$. The solution to the system is (3.38), (3.39) and

$$\left( \sigma_{S_t}^T + \sigma^T \right)^{-1} (\mu_{S_t} + D - rS_t) = \frac{A(w_t) r \alpha}{A(w_t) - F'(w_t) + r \alpha} b_t$$  \hspace{1cm} (A.45)

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and where

\[ b_t \equiv (\sigma s_t + \sigma) s + \sigma u \]

implying also

\[
Y_t^\top (\mu_t + D - r S_t) = \frac{A(w_t) (r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t
\]

\[
X_t^\top (\mu_t + D - r S_t) = \left( \frac{A(w_t) r\alpha (A(w_t) - F'(w_t))}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t - \frac{A(w_t) r\alpha}{A(w_t) - F'(w_t) + r\alpha} u^\top \sigma b_t \right)
\]

Using (A.45)-(3.39) and (A.42) we can rewrite (??) as

\[
0 = \rho - r - rF(w_t) + r \log(r) +
\]

\[
+ r\alpha \left( \frac{A(w_t) r\alpha (A(w_t) - F'(w_t))}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t - \frac{A(w_t) r\alpha}{A(w_t) - F'(w_t) + r\alpha} u^\top \sigma b_t \right) + u^\top D - \frac{1}{2} (r\alpha)^2 \frac{(A(w_t) - F'(w_t))^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t + F'(w_t) (w_t)(r w_t - q(w_t)^{-1}) w_t
\]

\[
+ \frac{A(w_t) (r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t + \frac{1}{2} \left[ F''(w_t) - F'(w_t)^2 \right] \frac{(r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t
\]

\[
- r\alpha F'(w_t) \frac{r\alpha (A(w_t) - F'(w_t))}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t.
\]

which simplifies to

\[
0 = \rho - r - rF(w_t) + r \log(r) + rau^\top D + F'(w_t)(r - q(w_t)^{-1}) w_t
\]

\[ (A.46) \]

\[
+ \frac{(r\alpha)^2}{2} \left( F'' + A(w_t)^2 \right) \frac{b_t^\top b_t}{(A(w_t) - F'(w_t) + r\alpha)^2} - \frac{A(w_t) (r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} u^\top \sigma b_t.
\]

Similarly, substituting (A.45)-(3.39) and (A.42) and the definition of \(A(w_t)\) into (A.39) and simplifying gives

\[
\rho q = \gamma q^{1-\frac{1}{\gamma} + \left( r - q^{-\frac{1}{\gamma}} \right) q w + rq(1-\gamma) + \frac{1}{2} \left( q'' + \frac{2q'\gamma}{w} - \frac{2q^2}{w^2} + q(1-\gamma)\gamma \right) \frac{(r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t^\top b_t.\]
Before, we simplify (A.46)-(A.47) further, we derive the ODE for $g(w_t)$. Note that from Ito’s Lemma

$$
\mu_{st} = \mu w_t S'(w_t) + \frac{1}{2} \sigma_{wt}^2 \sigma_{wt} S''(w_t) =
$$

$$
= (r w_t - q(w_t)) w_t^{-\frac{1}{2}} + \frac{A(w_t)(r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t \sigma_{t} + \frac{1}{2} \frac{(r\alpha)^2}{(A(w_t) - F'(w_t) + r\alpha)^2} b_t \sigma_{t}
$$

$$
\sigma_{st} = \sigma_{wt} S'(w_t) = (\sigma_s + \sigma) Y_t S'(w_t) =
$$

$$
= \frac{r\alpha}{A(w_t) - F'(w_t) + r\alpha} b_t \sigma_{t},
$$

and

$$
\sigma_{st} = \sigma_{wt} S'(w_t) = (\sigma_s + \sigma) Y_t S'(w_t) =
$$

$$
\sigma_{st} = \sigma_{wt} S'(w_t) = (\sigma_s + \sigma) Y_t S'(w_t) =
$$

$$
\sigma_{st} = \sigma_{wt} S'(w_t) = (\sigma_s + \sigma) Y_t S'(w_t) =
$$

Substituting back $\sigma_s$ into the definition of $b_t$, we get

$$
b_t = ((\sigma_s + \sigma) s + \sigma u) = \frac{r\alpha}{A(w_t) - F'(w_t) + r\alpha} b_t \sigma_{t} + \sigma (u + s)
$$

$$
b_t = \frac{\sigma (u + s)}{1 - \frac{r\alpha}{A(w_t) - F'(w_t) + r\alpha} b_t \sigma_{t} s},
$$

and using (1.3), we get (3.40) and (3.41). Substituting (3.40) and (3.41) into (A.45), we find the ODE for $g(w_t)$ (3.35). Also, given (3.40), (A.46) and (A.47) simplifies to (3.36) and (3.33). Using the same steps for the $\gamma = 1$ case, we get (3.34). Finally, using (3.40) and (3.41) and the definition of $E_t(dR_t)$ and (A.45), we get (3.37).

**Proof of Proposition 3.5:** From (1.3)

$$
\frac{\partial S_{nt}}{\partial u_n} = - (\alpha - g(w_t)) \Sigma_{nn}.
$$

For $\frac{\partial X_{nt}}{\partial u_n}$, we use (3.39),(A.45),(A.49) and (3.40) to rewrite

$$
(\sigma_s + \sigma) Y_t = \frac{r\alpha}{A(w_t) - F'(w_t) + r\alpha} b_t
$$
as

$$\sigma\left( r \alpha g'(w_t) - \frac{(u+s)(u+s)^\top \Sigma}{\left( A(w_t) - F'(w_t) + r\alpha - r \alpha g'(w)(u+s)^\top \Sigma^\top s \right)} + I \right) Y_t =$$

$$= r \alpha \frac{\sigma(u+s)}{\left( A(w_t) - F'(w_t) + r\alpha - r \alpha g'(w)(u+s)^\top \Sigma^\top s \right)}$$

multiplying both sides by $\sigma^{-1}$ shows that $Y_t$ must be collinear with $(u+s)$. So let us write $Y_t = (u+s)v$ giving

$$r \alpha g'(w_t) - \frac{(u+s)^\top \Sigma(u+s)v}{\left( A(w_t) - F'(w_t) + r\alpha - r \alpha g'(w)(u+s)^\top \Sigma^\top s \right)} + v =$$

$$= r \alpha \frac{1}{\left( A(w_t) - F'(w_t) + r\alpha - r \alpha g'(w)(u+s)^\top \Sigma^\top s \right)}$$

which implies

$$v = \frac{1}{g'(w_t)(u+s)^\top \Sigma u + \frac{A(w_t) - F'(w_t)}{r\alpha} + 1}.$$

Hence, by market clearing

$$\frac{\partial X_n}{\partial u_n} = -\frac{\partial Y_n}{\partial u_n} = \frac{1}{g'(w_t)(u+s)^\top \Sigma u + \frac{A(w_t) - F'(w_t)}{r\alpha} + 1}$$

giving the result.
References

Figure 1: Equilibrium objects with risk-neutral arbitrageurs ($\gamma = 0$) under two scenarios. In each scenario there are two risky assets with independent and identically distributed payoffs. Parameter values are $r = 0.02$, $\rho = 0.04$, $\sigma_1 = \sigma_2 = 0.1$, $(u + s)^\top \Sigma (u + s) = 0.0225$ and $s = 0$ (first scenario) and $s = 0.5$ (second scenario).
Figure 2: Equilibrium objects with log arbitrageurs ($\gamma = 1$) under two scenarios. In each scenario there are two risky assets with independent and identically distributed payoffs. Parameter values are $r = 0.02$, $\rho = 0.04$, $\sigma_1 = \sigma_2 = 0.1$, $(u + s)^\top \Sigma (u + s) = 0.0225$ and $s = 0$ (first scenario) and $s = 0.5$ (second scenario).
Figure 3: Equilibrium objects with risk-neutral arbitrageurs ($\gamma = 0$) under two scenarios. In each scenario there are two risky assets with independent and identically distributed payoffs. Parameter values are $r = 0.02$, $\rho = 0.04$, $\sigma_1 = \sigma_2 = 0.1$, $u^\top \Sigma u = 0.0225$ and $s = 0$. In the second scenario demand of hedgers is stochastic with $\kappa = 0.01$. 

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Figure 4: Equilibrium objects with log arbitrageurs ($\gamma = 1$) under two scenarios. In each scenario there are two risky assets with independent and identically distributed payoffs. Parameter values are $r = 0.02$, $\rho = 0.04$, $\sigma_1 = \sigma_2 = 0.1$, $u^\top \Sigma u = 0.0225$ and $s = 0$. In the second scenario demand of hedgers is stochastic with $\kappa = 0.01$. 

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Figure 5: Equilibrium objects with log-arbitrageurs ($\gamma = 1$) under two scenarios. In each scenario there are two risky assets with independent and identically distributed payoffs. In the first scenario hedgers are short-term mean-variance investors, while in the second one hedgers are infinitely lived CARA investors. Parameter values are $r = 0.02$, $\rho = 0.04$, $\sigma_1 = \sigma_2 = 0.1$, $(u+s)^\top \Sigma (u+s) = 0.0225$, $s = 0.5$ and $\alpha = 2$ (first scenario) and $\alpha = 2$ (second scenario).