Beauty Contest and Iterated Expectations in Asset Markets

Franklin Allen
Stephen Morris
Hyun Song Shin

Presented by
Orkhan Najafov
The Theory

• Keynesian Beauty Contest

• the theoretical literature on asset pricing has, until recently, failed to develop models that validate the role of higher order beliefs in asset pricing

• the purpose of this paper is to explain the role of higher order expectations in an asset pricing context
• higher order expectations do not make an appearance in standard competitive asset pricing models with a representative investor
• why asset prices in a competitive market may fail to reflect solely the discounted expected payoffs?
• the martingale property of asset prices
• the law of iterated expectations
• if there is a differential information between investors, average expectations fail to satisfy the law of iterated expectations

• if an individual has access to both private and public information about an asset’s payoffs, and they are of equal value in predicting the asset’s payoffs he will put more weight on the public signal than on the private signal
Background

- dynamic, multi-period (from 1 to T) asset pricing with a single risky asset $\theta$ that will be liquidated at date $T + 1$
- overlapping generations of traders who trade when young, and consume when old
- individual and public expectations satisfy the law of iterated expectations

\[
E_{it} (E_{i,t+1} (\theta)) = E_{it} (\theta)
\]
and
\[
E_{t}^* (E_{t+1}^* (\theta)) = E_{t}^* (\theta).
\]

- analogous property for average expectations will typically fail under asymmetric information

\[
\overline{E}_{t} (\overline{E}_{t+1} (\theta)) \neq \overline{E}_{t} (\theta)
\]
• $\theta$ is distributed normally with mean $\gamma$ and variance $1/\alpha$

• each player $i$ in a continuum observes a signal

\[ x_i = \theta + \epsilon_i. \]

• $\epsilon_i$ is distributed in the population with mean $0$ and variance $1/\beta$

\[ E_i(\theta) = \frac{\alpha \gamma + \beta x_i}{\alpha + \beta} \]

• because all the information is available at all dates we can drop the time subscript
• if we take the average we get the following

\[ \overline{E}(\theta) = \frac{\alpha y + \beta \theta}{\alpha + \beta} \]

• we take expectation and average it across individuals again

\[ E_i \left( \overline{E}(\theta) \right) = \frac{\alpha y + \beta E_i(\theta)}{\alpha + \beta} \]

\[ = \frac{\alpha y + \beta \left( \frac{\alpha y + \beta x_i}{\alpha + \beta} \right)}{\alpha + \beta} \]

\[ = \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y + \left( \frac{\beta}{\alpha + \beta} \right)^2 x_i \]

\[ \overline{E} \left( \overline{E}(\theta) \right) = \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y + \left( \frac{\beta}{\alpha + \beta} \right)^2 \theta \]
• iterating this operation we get
\[
\bar{E}^k (\theta) = \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^k\right) y + \left(\frac{\beta}{\alpha+\beta}\right)^k \theta
\]

• the expectation of expectation is biased towards the public signal \(y\)
\[
as k \to \infty, \bar{E}^k (\theta) \to y.
\]

• put back the time subscripts
\[
\bar{E}_t (\bar{E}_{t+1} (\theta)) = \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^2\right) y + \left(\frac{\beta}{\alpha+\beta}\right)^2 \theta \neq \frac{\alpha y + \beta \theta}{\alpha + \beta} = \bar{E}_t (\theta)
\]

• for any time period the expression would be
\[
\bar{E}_t (\bar{E}_{t+1} (\cdots \bar{E}_{T-2} (\bar{E}_{T-1} (\theta)))) = \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^{T-t}\right) y + \left(\frac{\beta}{\alpha+\beta}\right)^{T-t} \theta.
\]
• if asset has liquidation value $\theta$ at date $T$ and the asset is priced according to the asset pricing formula

$$p_t = \mathbb{E}_t (p_{t+1})$$

• then we have

$$p_t = \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^{T-t}\right) y + \left(\frac{\beta}{\alpha+\beta}\right)^{T-t} \theta.$$

• the more trading periods there are, the higher the variance of the price
Asset Prices

• learning from prices
• liquidation value of $\theta$ is determined at date 1 and it remains fixed until the asset is liquidated at $T+1$
• overlapping generations of traders born at each date $t$ indexed by the unit interval $[0,1]$
• at any trading date $t$ there is a unit mass of young traders and unit mass of old traders
• each new trader do not know the true value of the fundamental, but everybody knows full history of past and current prices and realization of private signal

$$x_{it} = \theta + \varepsilon_{it}$$
Figure 1: Path of fundamental value

Figure 2: Private information
• traders have exponential utility function
  \[ u(c) = -e^{-\frac{c}{\tau}} \]

• \( \tau \) is the reciprocal of absolute risk aversion or traders’ risk tolerance

• noisy supply of assets, \( s_t \), distributed normally with mean \( \theta \) and precision \( \gamma_t \)

• the traders’ demand for assets
  \[ \frac{\tau}{\text{Var}_i T(\theta)} (E_{iT}(\theta) - p_T) \]

• conditional variance of fundamental is identical across traders, therefore aggregate demand demand at date \( T \) is
  \[ \frac{\tau}{\text{Var}_T(\theta)} (\bar{E}_T(\theta) - p_T) \]
• from market clearing condition we get

\[ p_T = \bar{E}_T(\theta) - \frac{\text{Var}_T(\theta)}{\tau} s_T \]

\[ p_{T-1} = \bar{E}_{T-1}(p_T) - \frac{\text{Var}_{T-1}(p_T)}{\tau} s_{T-1} \]

• by substituting for price at date T we get

\[ p_{T-1} = \bar{E}_{T-1} \bar{E}_T(\theta) - \frac{\text{Var}_{T-1}(p_T)}{\tau} s_{T-1} \]

• if we continue iterating further back in time

\[ p_t = \bar{E}_t \bar{E}_{t+1} \cdots \bar{E}_T(\theta) - \frac{\text{Var}_t(p_{t+1})}{\tau} s_t \]

• the price at date t is the average expectation at date t of the average expectation at the next trading date, etc. of the final liquidation value \( \theta \)
Results

• two essential questions of the paper
  – Is there any systematic difference between the price $p_t$ and the consensus value of the fundamentals given by $E_t(\theta)$?
  – How quickly does the price adjust to the shift in fundamentals at date 1?
Figure 3: Mean of time paths of $p_t$ and $\overline{E}_t(\theta)$
Proposition 1  For all \( t < T \)

\[
E_s(|p_t - \theta|) > E_s\left(\left|\bar{E}_t(\theta) - \theta\right|\right)
\]

It is only at the final trading date \( T \), that we have \( E_s(p_T) = E_s(\bar{E}_T(\theta)) \)

Proposition 2  There exist weights \( \{\lambda_t\} \) with \( 0 < \lambda_T < \lambda_{T-1} < \cdots < \lambda_2 < \lambda_1 < 1 \) such that

\[
E_s(p_t) = \lambda_ty + (1 - \lambda_t)\theta
\]

Proposition 3  Let \( q_t \) be defined as

\[
q_t = \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^{T-t+1}\right)y + \left(\frac{\beta}{\alpha + \beta}\right)^{T-t+1}\theta
\]

Then, as \( \tau \to 0 \), we have

\[
E_s(p_t) \to q_t
\]