

## Equivalence of detrending and including a time trend as an explanatory variable

Let us suppose we estimate the following model by OLS:

$$y_t = \alpha + \beta x_t + \delta t + u_t$$

We get the estimated coefficients by solving the linear system of equations:

$$\sum_{t=1}^T (y_t - \hat{\alpha} - \hat{\beta}x_t - \hat{\delta}t) = 0 \quad (1)$$

$$\sum_{t=1}^T x_t (y_t - \hat{\alpha} - \hat{\beta}x_t - \hat{\delta}t) = 0 \quad (2)$$

$$\sum_{t=1}^T t (y_t - \hat{\alpha} - \hat{\beta}x_t - \hat{\delta}t) = 0 \quad (3)$$

Alternatively, let us first detrend  $y_t$  and  $x_t$  by running the regressions:

$$y_t = \alpha_y + \delta_y t + \varepsilon_t^y$$

and

$$x_t = \alpha_x + \delta_x t + \varepsilon_t^x.$$

We can estimate the coefficients of these detrending equations by solving the following two sets of linear equations:

$$\sum_{t=1}^T \ddot{y}_t \equiv \sum_{t=1}^T (y_t - \hat{\alpha}_y - \hat{\delta}_y t) = 0$$

$$\sum_{t=1}^T t \ddot{y}_t \equiv \sum_{t=1}^T t (y_t - \hat{\alpha}_y - \hat{\delta}_y t) = 0$$

and

$$\sum_{t=1}^T \ddot{x}_t \equiv \sum_{t=1}^T (x_t - \hat{\alpha}_x - \hat{\delta}_x t) = 0 \quad (4)$$

$$\sum_{t=1}^T t \ddot{x}_t \equiv \sum_{t=1}^T t (x_t - \hat{\alpha}_x - \hat{\delta}_x t) = 0. \quad (5)$$

Denote the detrended residuals by  $\ddot{x}_t \equiv x_t - \hat{\alpha}_x - \hat{\delta}_x t$  and  $\ddot{y}_t \equiv y_t - \hat{\alpha}_y - \hat{\delta}_y t$ . Now gradually substitute the fitted values plus residuals instead of  $y_t$  and  $x_t$  into (2). For notational brevity, I will use the residual form  $\hat{u}_t \equiv y_t - \hat{\alpha} - \hat{\beta}x_t - \hat{\delta}t$  where convenient.

$$0 = \sum_{t=1}^T x_t \hat{u}_t = \sum_{t=1}^T (\ddot{x}_t + \hat{\alpha}_x + \hat{\delta}_x t) \hat{u}_t = \sum_{t=1}^T \ddot{x}_t \hat{u}_t + \hat{\alpha}_x \sum_{t=1}^T \hat{u}_t + \hat{\delta}_x \sum_{t=1}^T t \hat{u}_t = \sum_{t=1}^T \ddot{x}_t \hat{u}_t$$

To get the last equality, I applied (1) and (3). Let us continue by substituting the fitted values plus detrended residuals into  $\hat{u}_t$ :

$$0 = \sum_{t=1}^T \ddot{x}_t \left( y_t - \hat{\alpha} - \hat{\beta}x_t - \hat{\delta}t \right) = \sum_{t=1}^T \ddot{x}_t \left[ \ddot{y}_t + \hat{\alpha}_y + \hat{\delta}_y t - \hat{\alpha} - \hat{\beta} \left( \ddot{x}_t + \hat{\alpha}_x + \hat{\delta}_x t \right) - \hat{\delta}t \right] \quad (6)$$

Rearranging yields:

$$0 = \sum_{t=1}^T \ddot{x}_t \left( \ddot{y}_t - \hat{\beta}\ddot{x}_t \right) + \sum_{t=1}^T \ddot{x}_t \left[ \hat{\alpha}_y + \hat{\delta}_y t - \hat{\alpha} - \hat{\beta}\hat{\alpha}_x - \hat{\beta}\hat{\delta}_x t - \hat{\delta}t \right]$$

Now notice that the second term on the right hand side is nothing but a linear combination of first order conditions (4) and (5):

$$\begin{aligned} & \sum_{t=1}^T \ddot{x}_t \left[ \hat{\alpha}_y + \hat{\delta}_y t - \hat{\alpha} - \hat{\beta}\hat{\alpha}_x - \hat{\beta}\hat{\delta}_x t - \hat{\delta}t \right] = \\ & = \left( \hat{\alpha}_y - \hat{\alpha} - \hat{\beta}\hat{\alpha}_x \right) \sum_{t=1}^T \ddot{x}_t + \left( \hat{\delta}_y - \hat{\beta}\hat{\delta}_x - \hat{\delta} \right) \sum_{t=1}^T t\ddot{x}_t = 0 \end{aligned}$$

Therefore:

$$\sum_{t=1}^T \ddot{x}_t \left( \ddot{y}_t - \hat{\beta}\ddot{x}_t \right) = 0$$

and

$$\hat{\beta} = \frac{\sum_{t=1}^T \ddot{x}_t \ddot{y}_t}{\sum_{t=1}^T \ddot{x}_t^2}$$

which means that  $\hat{\beta}$  can equivalently be computed by running a simple regression of the detrended dependent variable  $\ddot{y}_t$  on the detrended explanatory variable  $\ddot{x}_t$  without a constant.<sup>1</sup> The important message of the derivation is that the two methods are not only consistent for the same  $\beta$  but also yield the exact same estimate of  $\hat{\beta}$ .<sup>2</sup>

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<sup>1</sup>Interestingly, we can get the same estimate  $\hat{\beta}$  even if we only detrend the explanatory variable  $x_t$ , and run a regression of the non-detrended dependent variable  $y_t$  on the detrended version of the explanatory variable  $\ddot{x}_t$ . Can you show how to get this result? (Hint: have a look at (6). What happens if we only substitute the fitted values plus residuals for  $x_t$ , but leave  $y_t$  as it is?)

<sup>2</sup>To check your understanding of the derivation, you might want to try your hand at generalizing the result to more than one explanatory variable (besides the time trend).