Normal-normal regression

$$\mathbf{x} \equiv \begin{pmatrix} y & x \end{pmatrix}' \sim N(\mu, \Sigma)$$
$$\boldsymbol{\mu} = \begin{pmatrix} \mu_y & \mu_x \end{pmatrix}'$$
$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_y^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_x^2 \end{bmatrix}$$

Joint distribution:

$$\begin{aligned} f(y,x) &= \frac{1}{2\pi}|\Sigma|^{-1/2}\exp\left\{-\frac{1}{2}\left[(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]\right\} \\ &= \frac{1}{2\pi\sqrt{\sigma_y^2\sigma_x^2-\sigma_{xy}^2}}\exp\left\{-\frac{1}{2(\sigma_y^2\sigma_x^2-\sigma_{xy}^2)}\left[\begin{pmatrix} y-\mu_y & x-\mu_x \end{pmatrix}\begin{bmatrix} \sigma_x^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_y^2 \end{bmatrix}\begin{pmatrix} y-\mu_y \\ x-\mu_x \end{pmatrix}\right]\right\} \\ &= \frac{1}{2\pi\sqrt{\sigma_y^2\sigma_x^2-\sigma_{xy}^2}}\exp\left\{-\frac{(y-\mu_y)^2\sigma_x^2-2(x-\mu_x)(y-\mu_y)\sigma_{xy}+(x-\mu_x)^2\sigma_y^2}{2(\sigma_y^2\sigma_x^2-\sigma_{xy}^2)}\right\} \end{aligned}$$

Marginal distribution of x:

$$f(x) = N(\mu_x, \sigma_x^2) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\}$$

Conditional distribution of y given x:

$$\begin{aligned} f(y|x) &= \frac{f(y,x)}{f(x)} = \frac{\sqrt{2\pi}\,\sigma_x}{2\pi\sqrt{\sigma_y^2\sigma_x^2-\sigma_{xy}^2}}\exp\left\{-\frac{(y-\mu_y)^2\sigma_x^2+(x-\mu_x)^2\sigma_y^2-2(x-\mu_x)(y-\mu_y)\sigma_{xy}}{2\sigma_x^2\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}+\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \\ &= \frac{1}{\sqrt{2\pi\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}}\exp\left\{-\frac{(y-\mu_y)^2\sigma_x^2+(x-\mu_x)^2\sigma_y^2-2(x-\mu_x)(y-\mu_y)\sigma_{xy}-(x-\mu_x)^2\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}{2\sigma_x^2\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}\right\} \\ &= \frac{1}{\sqrt{2\pi\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}}\exp\left\{-\frac{(y-\mu_y)^2-2(x-\mu_x)(y-\mu_y)\frac{\sigma_{xy}}{\sigma_x^2}+(x-\mu_x)^2\frac{\sigma_{xy}^2}{\sigma_x^4}}{2\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}\right\} \\ &= \frac{1}{\sqrt{2\pi\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}}\exp\left\{-\frac{\left[(y-\mu_y)-(x-\mu_x)\frac{\sigma_{xy}}{\sigma_x^2}\right]^2}{2\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}\right\} = \frac{1}{\sqrt{2\pi\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}}\exp\left\{-\frac{\left[y-\left(\mu_y-\mu_x\frac{\sigma_{xy}}{\sigma_x^2}+x\frac{\sigma_{xy}}{\sigma_x^2}\right)\right]^2}{2\left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)}\right\} \end{aligned}$$

or, in other words,

$$y|x \sim N\left[\mu_y-\frac{\sigma_{yx}}{\sigma_x^2}\mu_x+\frac{\sigma_{yx}}{\sigma_x^2}x, \left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)\right]$$

or, more simply,

$$y|x \sim N\left[(\beta_0+\beta_1 x), \left(\sigma_y^2-\frac{\sigma_{xy}^2}{\sigma_x^2}\right)\right]$$

where

$$\beta_0 = \mu_y-\beta_1\mu_x$$
$$\beta_1 = \frac{\sigma_{yx}}{\sigma_x^2}$$

So

$$E(y|x) = \beta_0+\beta_1 x$$

exactly (not as an aprroximation)