Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models*

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Abstract

I derive simple conditions for the existence of a stable invariant distribution of an increasing Markov process on a non-compact state space. I use these conditions in two workhorse economic models. First, in my main contribution, I settle a conjecture of Carroll by characterizing the conditions under which the canonical buffer stock saving model has a stable invariant distribution. Second, I show that in the Brock-Mirman one-sector growth model, a stable invariant distribution exists for a large class of production functions. These results help characterize long term behavior in dynamic economies, and have implications for the validity of numerical solutions.

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In many stochastic dynamic models in economics, agents' behavior follows a Markov process. In such models, due to unremitting uncertainty the economy never settles down to a deterministic "rest point." Therefore it is often more convenient to consider a rest point in the stochastic sense, i.e., a distribution to which the economy eventually converges for any initial condition. Such a stable distribution, if exists, can be viewed as the long run equilibrium in a stochastic dynamic model.

This paper studies the existence of stable distributions for a class of Markov processes defined on a non-compact state space. Allowing the state space to be non-compact is important for many economic applications. For example, consider the dynamic macroeconomic models used in consumption theory, economic growth and asset pricing. In many of these models, consumers have power utility preferences and shocks are lognormally distributed, a combination that by construction leads to an unbounded state space.

Determining whether a stable distribution exists in these non-compact models is useful both for theoretical and applied reasons. From a theoretical perspective, such results help characterize whether long term behavior is stationary or explosive, which in turn has further economic implications. For an example, consider the main application of this paper, the buffer stock saving model analyzed in more detail below. When this model has a stable invariant distribution, consumption is mean reverting, and hence buffer stock behavior obtains (as in Carroll, 1997); in particular, consumption growth is predictable and excessively sensitive to temporary shocks. In contrast, when there is no stable invariant distribution, eventually consumption becomes a random walk (as in Hall, 1978), and therefore, in the long run, the standard permanent income hypothesis holds: consumption is unpredictable and not excessively sensitive. Thus, in this model, whether a stable invariant distribution exists has strong implications for long term consumption behavior.

From an applied perspective, stable distributions can matter for the validity of numerical solutions. In practice, numerically solving macroeconomic models requires imposing bounds on the realizations of shocks, resulting in a compact state space. For these numerical predictions to approximate the original model, the simulated and the original economies should have similar long term behavior. Typically, compactness implies that a stable invariant distribution exists in the numerically solved economy; hence numerical predictions are likely to be misleading if the original model leads to non-stationary behavior. In this case, the seemingly stationary dynamics in the simulations also depend on the way bounds are imposed on the state space, and therefore predictions are partly determined by numerical details and not by the underlying economic content. In these circumstances, theoretical results about invariant distributions help identify the "right" numerical model that matches long term behavior.

In this paper, I study the existence of stable distributions for Markov processes that are increasing in a probabilistic sense: loosely when the process is started from a higher initial value, it assumes large values with higher probability. In an influential paper, Hopenhayn and Prescott (1992) studied the existence and stability of invariant distributions for increasing Markov processes on a compact state space. In this paper I establish analogous results for non-compact spaces; develop conditions allowing to check for stable invariant distributions relatively easily in applications; and then use these conditions to characterize stable invariant distributions in buffer-stock saving and stochastic growth models.

In Section 1 of the paper, I show that an increasing Markov process has a stable distribution if two conditions hold. The first of these, which I call uniform asymptotic tightness, is a new condition developed in this paper. Intuitively, this condition ensures that the process does not escape to infinity in a probabilistic sense, and hence serves as a replacement of the compactness requirement in earlier work. In particular, uniform asymptotic tightness is automatically satisfied if the state space is compact. My second sufficient condition is a weak mixing requirement, which relaxes the mixing condition of Hopenhayn and Prescott (1992). Because the processes I consider need not be continuous, stability does not imply invariance; however, I also establish simple conditions under which the stable distributions I find are also invariant. Moreover, I show that, except for a small set of increasing Markov processes, the existence of a stable invariant distribution implies both asymptotic tightness and weak mixing, and hence these conditions are not only sufficient, but also "almost necessary."

Also in Section 1, I develop simple conditions to verify uniform asymptotic tightness in applications. These conditions are based on the idea that a random walk with a negative drift and a lower bound will not escape to infinity; and that any process which is bounded by such a random walk is thus asymptotically uniformly tight. The applicability of these conditions comes from the fact that in many stochastic dynamic problems in economics, optimal choices evolve according to random walks (Hall, 1978), or, in the presence of constraints, can be bounded by such random walks. My conditions are also relatively easy to check, as they only require verifying that the mean of innovations is negative.

Section 2 presents two applications. First, in my main application, I consider a buffer-stock saving model similar to Carroll (1997) and Haliassos and Michaelides (2003).[1] Carroll has conjectured that under some conditions a stable invariant distribution exists in these models, but except for some restrictive special cases (Clarida 1987, Schechtman and Escudero 1976), the result has not been proved.[2] This is in part because the state space in these models is not compact, and hence results like Hopenhayn and Prescott's do not apply. Using the theoretical tools described above, I provide an essentially complete characterization of the parameters for which a stable invariant distribution exists. Loosely, I find that a stable invariant distribution exists if and only if consumption growth in an auxiliary economy with no labor income is lower than labor income growth in the original model. Intuitively, when the consumption of the buffer stock agent is high relative to his current income, his behavior is well-approximated by the auxiliary model with no labor income. If consumption growth in the auxiliary model is smaller then income growth in the buffer stock model, then the consumption to income ratio is expected to fall, leading to mean-reversion. In contrast, if the opposite inequality holds then consumption is expected to increase further relative to income, and in the long term consumption follows a random walk.

In practice, comparing these two growth rates is a relatively simple task, because the auxiliary model is a version of the Merton consumption problem, which has a known solution. In the special case when the buffer stock consumer can only invest in a safe financial asset (as in Carroll, 1997), the result implies that an invariant distribution always exists under the standard impatience requirement imposed in the literature to ensure that the consumer's problem has a solution. In contrast, when there is a risky investment opportunity as well, as in Haliassos and Michaelides (2003), the condition required for a stable invariant distribution is more stringent than the standard impatience condition, and hence there are cases where the model has a solution and yet no invariant distribution exists.

To see the implications of these results for consumption behavior, note that when a stable invariant distribution exists, the consumption to income ratio is mean reverting. It follows that high values of consumption per income must predict lower subsequent consumption growth, to bring down this ratio to its long term steady state. In particular, a positive temporary shock that raises the consumption to income ratio must lead to lower consumption growth in the future, i.e.,

[1] There is a large literature on buffer stock savings models, including Deaton (1991), Carroll (1997, 2004), Gourinchas and Parker (2002), Haliassos and Michaelides (2003) and Ludvigson and Michaelides (2001) among others.

[2] Clarida (1987) derives the result for bounded utility and marginal utility (which excludes the use of constant relative risk aversion utility), and no permanent shocks. Schechtman and Escudero (1976) focus on a case where the state space is compact. See also Deaton and Laroque (1992) for a related model of commodity prices, where an invariant distribution exists also by compactness.

the immediate consumption response to the shock is "too large." Thus, when a stable invariant distribution exists, consumption growth is predictable and displays excess sensitivity to temporary shocks, both of which are key features of buffer stock behavior. In contrast, for the parameters where no invariant distribution exists, behavior converges to the solution of the Merton consumption problem and hence consumption becomes an unpredictable random walk with no excess sensitivity, as in the standard permanent income hypothesis. The fact that with a risky asset, the model may generate either buffer stock behavior or a random walk emphasizes the importance of determining whether an invariant distribution exists, and calls for caution in simulations with truncated shocks that necessarily yield stationary behavior.

The results about a stable invariant distribution also have some direct implications for applied work. Carroll (1997, 2004) argues that when a stable invariant distribution exists, the growth rates of aggregate consumption and income are equal, which helps explain the "consumption/income parallel" of Carroll and Summers (1991) and has implications for estimating Euler equations in practice. A stable invariant distribution is also useful for computational reasons: Haliassos and Michaelides use the invariant wealth distribution to compute time-series and population averages of endogenous variables such as consumption.

As a second illustrative application of the theoretical results, I consider the one-sector growth model of Brock and Mirman (1972). The existence and stability of an invariant distribution was analyzed by Brock and Mirman in their original paper, and in much subsequent work.[3] This research generally focuses on the case where the capital stock is limited to a bounded interval, and where technology shocks are bounded both above and below. More recently, Stachurski (2002), Kamihigashi (2007) and Zhang (2007) derive the existence and stability of an invariant distribution with unbounded shock and capital stock, under the assumption that either the technology shocks enter the production function multiplicatively or the shock distribution is absolutely continuous. Using the tools developed in this paper, I establish the existence of a stable invariant distribution for log-supermodular production functions, under weaker assumptions about the distribution of the shock than these papers.

This paper builds on and contributes to the literature on invariant distributions of Markov processes. The current paper is a revised version of my 2008 working paper, which established the results on stability, but stated, erroneously, that that my proof of stability implies invariance

[3] See for example Mirman (1973), Mirman and Zilcha (1975) , Razin and Yahav (1979), Stokey, Lucas and Prescott (1989) and Hopenhayn and Prescott (1992).

without additional conditions.[4] My current draft fixes this mistake, but otherwise is very similar to the previous version. In a very nice working paper, Kamihigashi and Stachurski (2012)—written subsequent to, but indendently of my 2008 draft—establish existence and stability results under slightly weaker conditions than mine, on the way toward their main contribution, the study of ergodic properties. Relative to their work, the main contributions of the present paper lie in developing conditions for stability which are easy to verify in applications, and in the buffer-stock savings application.

Also related is the work of Bhattacharya and Lee (1988), which develops existence and uniqueness results for increasing Markov processes on non-compact state spaces. Their results make use of a fairly strong mixing condition which is not satisfied in the two applications considered in this paper, and which is unlikely to hold in many economic applications. Torres (1990) explores the existence of invariant distributions in non-compact state spaces, but does not study stability.

1 Invariant Distributions of Markov Processes

1.1 Preliminaries

My goal is to study the stable distributions of Markov processes that take on potentially unbounded values. Let $C \subseteq \mathbb{R}^N$ be a closed but not necessarily compact set, and consider a Markov process x_t, $t = 0, 1, 2, \ldots$, that assumes values in C. The dynamic of this process is characterized by a transition function $P : C \times \mathcal{B} \to [0,1]$, where $\mathcal{B}$ denotes the collection of Borel sets in $\mathbb{R}^N$. The transition function is defined such that for each $x \in C$ and $A \in \mathcal{B}$, the conditional probability that $x_{t+1} \in A$ given that $x_t = x$ is $P(x, A)$. I assume that for all $x \in C$, $P(x, .)$ is a probability measure that satisfies $P(x, C) = 1$, and that for all $A \in \mathcal{B}$, $P(., A)$ is a measurable function. The n-step transition function $P^n(x, A)$ is defined analogously to be the conditional probability that $x_{t+n} \in A$ given that $x_t = x$.

For any probability measure μ over C, define the measure $T^*\mu$ as

$$(T^*\mu)(A) = \int_C P(z, A)\mu(dz). \tag{1}$$

With this notation, the Markov process induces a transition operator T^* on the set of probability measures over C. Intuitively, $(T^*\mu)(A)$ is the probability that next period the process is in the set

[4] I thank John Stachurski for pointing out that in my previous draft invariance follows only under the Feller property.

A provided that this period its value is drawn according to μ.[5]

The process P is Feller if for any sequence $x^n \to x$ in $\mathbb{R}^N$, the probability distributions $P(x^n, .)$ converge weakly to $P(x, .)$.[6] Intuitively, small variation in the state today induces small variation in the conditional distribution of the state tomorrow.

A probability distribution μ on C is an invariant distribution of the process x_t if $T^*\mu = \mu$. The process x_t started from an initial distribution λ converges to a distribution μ if the sequence of probability measures $T^{*n}\lambda$ for $n = 1, 2, 3,...$ converges weakly to μ. A distribution μ is stable if the process converges to μ for any initial distribution λ on C. Note, in this definition we do not require that the stable distribution itself be invariant. However, if the Markov process is Feller, it is easy to see that a stable distribution has to be invariant. Also, note that, by definition, a stable invariant distribution, if it exists, must be unique.

For two probability distributions μ and μ' on $\mathbb{R}^N$, write $\mu \geq \mu'$ if μ dominates μ' in the sense of first order stochastic dominance, that is, if

$$\int_{\mathbb{R}^N} f d\mu \geq \int_{\mathbb{R}^N} f d\mu'$$

holds for all bounded, increasing functions $f : \mathbb{R}^N \to \mathbb{R}$. The Markov process associated with the transition function P is *increasing* if for all x and x' in C, $x \geq x'$ implies $P(x, .) \geq P(x', .)$.

In the analysis below, I assume for simplicity that the set C on which the Markov process is defined is an "interval set" of the form $[a, b]$, $[a, b)$, $(a, b]$ or (a, b), where a and b are two N-dimensional vectors with coordinates that are permitted to be infinite. Here $[a, b] = \{x \in \mathbb{R}^N | a \leq x \leq b\}$, $[a, b) = \{x \in \mathbb{R}^N | a \leq x < b\}$ and $(a, b]$ and (a, b) are defined analogously. Interval sets commonly used in economic applications include $\mathbb{R}$ and $[0, \infty)$.[7]

I say that the process P has a probabilistic lower bound if there exists a distribution $\underline{\lambda}$ such that $T^*\underline{\lambda} \geq \underline{\lambda}$. Similarly, the process has a probabilistic upper bound if there exists a distribution $\overline{\lambda}$ such that $T^*\overline{\lambda} \leq \overline{\lambda}$. Using the notation that δ_x stands for the probability distribution concentrated on x, it is easy to see that when C is a compact interval set $[a, b]$, $\underline{\lambda} = \delta_a$ and $\overline{\lambda} = \delta_b$ serve as probabilistic lower and upper bounds. As I show in the applications, such bounds often exist even

[5] I follow the literature in denoting this operator by T^* (see e.g., Stokey, Lucas and Prescott, 1989). The reason for the star superscript is that T^* is the adjoint of an operator T which is introduced in the appendix.

[6] A sequence of probability measures μ_j are weakly convergent with limit μ iff for all bounded continuous functions f, $\lim_{j\to\infty} \int f d\mu_j = \int f d\mu$.

[7] For a Markov process defined on a non-interval set, one can proceed by first extending the process to an interval set and then applying the results of this paper. The extension is often trivial, thus focusing on interval sets is a small restriction in practice.

if the more interesting case when the state space is not compact.

1.2 Existence of a Stable Distribution

This section presents sufficient conditions for the existence of a stable distribution for increasing Markov processes. Stating the result requires two definitions.

Definition 1 *A Markov process with transition function $P(x, A)$ satisfies uniform asymptotic tightness if for all $\delta > 0$ there exists a compact set $C_\delta \subseteq C$ such that* $\lim_{n\to\infty} \inf P^n(x, C_\delta) > 1 - \delta$ *for all $x \in C$.*

To understand the definition, recall the concept of tightness from probability theory (e.g., Billingsley, 1995): a sequence of probability measures μ_n is tight if for all $\delta > 0$, there exists a compact set C_δ such that $\mu_n(C_\delta) > 1 - \delta$ for all n. Intuitively, tightness is a condition that prevents the mass of the probability distributions μ_n from "escaping to infinity" (Billingsley, 1995). Uniform asymptotic tightness requires that the sequence of probability measures $P^n(x, .)$ be tight, and moreover that the tightness condition holds with the same C_δ sets for all x as $n \to \infty$. This condition thus ensures that the dynamics of the Markov process is prevented from escaping to infinity uniformly across all initial conditions x. In the special case when P has probabilistic upper and lower bounds (e.g., when C is compact), uniform asymptotic tightness is automatically satisfied.

As we will see below, uniform asymptotic tightness is not only part of a set of sufficient conditions, but is also necessary for a Markov process to have a stable invariant distribution. Intuitively, when a stable invariant distribution exists, the process will converge to that distribution for all initial conditions, and hence for n large, a large part of its mass will be in the region where most of the invariant distribution is concentrated.

Definition 2 *A Markov process with transition function $P(x, A)$ satisfies weak mixing if there exists $c \in C$ with the property that for all $x \in C$ there are $\underline{S}$ and $\bar{S}$ positive integers (which may depend on x) such that*

$$P^{\underline{S}}(x, (c, \infty)) > 0 \tag{2}$$

and

$$P^{\bar{S}}(x, (-\infty, c)) > 0. \tag{3}$$

Mixing conditions are often used to establish the existence of a stable invariant distribution. Intuitively, mixing rules out the possibility of two invariant distributions, one concentrated above c, the other concentrated below c. The logic is simple: if an invariant distribution is fully concentrated below c, then for any x in the support of that distribution, condition (2) must fail.

The condition in Definition 2 is related to several mixing conditions in the literature. Bhattacharya and Lee (1988) impose a similar mixing assumption, but crucially, require mixing to be *uniform*; that is, that there exists $\varepsilon > 0$ such that $P^{\underline{S}}(x, (c, \infty)) > \varepsilon$ holds *for all* x.[8] Thus they require a universal $\underline{S}$ that works for all initial conditions x. In contrast, weak mixing only requires that for each x there exists an $\underline{S}$ such that $P^{\underline{S}}(x, (c, \infty)) > 0$, i.e., the number of steps required can depend on the initial condition. This distinction is important: in the economic applications I consider below, weak mixing can easily be verified, but the Bhattacharya-Lee mixing condition in general does not hold. An elegant mixing condition introduced by Kamihigashi Stachurski (2012b) is an order-reversing condition, which requires that when the process is started from two different initial values $x_L < x_H$, it will eventually reverse order with probability one. While my sense is that for a class of processes their mixing condition generalizes weak mixing, mine seems easier to verify in applications. Weak mixing also generalizes the strong mixing condition imposed by Hopenhayn and Prescott (1992), who, just like Bhattacharya and Lee, assume that $\underline{S} = \overline{S}$. Because Hopenhayn and Prescott work with compact sets C that contain a lower bound a and an upper bound b, they only need to impose mixing for $x = a$ and $x = b$; this implies mixing for all other $x \in C$ because the process is increasing. In our case, C need not be compact, and hence we require mixing for all $x \in C$.

As Dubins and Freedman (1966) noted, when C is an interval of $\mathbb{R}$, mixing is not only sufficient but also a necessary condition for the existence of a stable invariant distribution, except in the case when that distribution is concentrated on a single value. In higher dimensions there are examples where a stable invariant distribution exists but mixing is violated. However, as I show below, in "most cases" weak mixing is implied by a stable invariant distribution. The following result then provides both sufficient and "almost necessary" conditions for the existence of stable and invariant distributions for increasing Markov processes.

Theorem 1 *The following are true.*

1. If an increasing Markov process satisfies weak mixing and uniform asymptotic tightness, then it has a stable distribution.

2. If, in addition, the process is either Feller or has probabilistic upper and lower bounds, then the stable distribution is invariant.

[8] They also impose a second uniform mixing condition corresponding to (3).

3. If a Markov process has a stable distribution which is non-singular with respect to the Lebesgue measure, then it satisfies weak mixing and uniform asymptotic tightness.

The theorem extents Hopenhayn and Prescott's (1992) result about stable distributions for potentially non-compact state spaces. When C is a compact interval set, uniform asymptotic tightness is automatically satisfied, and Hopenhayn and Prescott's sufficient conditions obtain as a special case.

The difficult part of the result is part 1: that the conditions are sufficient for a stable distribution. The proof is technical and given in the appendix. To illustrate the logic, it is helpful to briefly review the argument of Hopenhayn and Prescott for the compact case. Let $\underline{k} < c < \overline{k}$, and assume momentarily that $\underline{k}$ is a lower bound and $\overline{k}$ is an upper bound of the set C. By monotonicity, the process started from $\overline{k}$ will dominate the process started from $\underline{k}$. However, mixing implies that eventually, ε of the probability mass of the process started from $\underline{k}$ will be above c, while ε of the mass when started from $\overline{k}$ will be below c. This implies that ε of the masses of the two processes reverse order. In addition, by compactness, the remainder $1-\varepsilon$ of the mass of both processes remain the $\left[\underline{k}, \overline{k}\right]$ interval. But then we can repeat the above logic to show that ε of these remainder masses will also reverse order as n becomes large. Repeating this "mixing logic" many times implies that eventually, the distributions $P^n\left(\underline{k}, .\right)$ and $P^n\left(\overline{k}, .\right)$ completely reverse order, which is only possible if they converge to the same limit.

A key step above is that the remainder $1-\varepsilon$ of the mass of $P^n\left(\underline{k}, .\right)$ remains above $\underline{k}$, so that the mixing logic can be repeated. This step is automatic if C is compact and $\underline{k}$ is its lower bound, but fails when C is not compact. In this case, while ε of the mass eventually migrates above c, part or all of the remainder $1-\varepsilon$ may go below $\underline{k}$, where the probability that it comes back above c can be much smaller.[9] This is a problem, because now we may only be able to repeat the mixing argument with a smaller ε, and infinite repetition with a sequence of decreasing ε values need not guarantee that the masses completely reverse order.

I deal with this problem by developing a bound on the measure that $P^n\left(\underline{k}, .\right)$ assigns to the complement of $[\underline{k}, \infty)$. The assumption of uniform asymptotic tightness is important for this, because it restricts the tail probabilities of $P^n\left(\underline{k}, .\right)$ for n large. The formal statement of this bound, expressed in terms of expected values of functions of the Markov process, is contained in

[9] Except if mixing is uniform, as in Bhattacharya and Lee (1988). Then, even if the process has fallen below $\underline{k}$, the probability that it comes back above c is at least ε independently of its current value. This illustrates the strength of their uniform mixing condition.

Lemma 1 in the Appendix. This bound in itself is not sufficient to conclude the proof, because repeating the mixing logic many times cumulates the error term in the bound. To avoid this difficulty, I choose the points $\underline{k}$ and $\overline{k}$ together with the number of repetitions to ensure that the cumulative error term does not blow up, and take the limits $n \to \infty$, $\underline{k} \to -\infty$ and $\overline{k} \to \infty$ in a carefully chosen order. This way I obtain limit results which are expressed in terms of the expected values of families of functions. To conclude the proof, I convert these results to statements about limits of probability measures.

1.3 Verifying Uniform Asymptotic Tightness

Theorem 1 is useful only to the extent that its sufficient conditions can be verified in applications. Two of the theorem's conditions, mixing and monotonicity, are well-studied and can be verified using standard tools, as in Hopenhayn and Prescott (1992). I now present results that can be used to verify the third condition, uniform asymptotic tightness.

Asymptotic tightness can be loosely interpreted as a requirement that the process does not escape to infinity. One way to ensure this is to assume that the mean of the process does not escape to infinity. This motivates the following definition.

Definition 3 *A process $P(x, A)$ has asymptotically bounded mean if there exists $K > 0$ such that for all $x \in C$, $\lim_{n\to\infty} \sup E|T^{*n}\delta_x| < K$.*

Because δ_x assigns unit mass to x, $\mathrm{E}|T^{*n}\delta_x|$ is the expected absolute value of the process after n periods, if started from x. Asymptotically bounded mean thus requires that the expected absolute value of the process eventually becomes bounded, irrespective of the initial value.

Proposition 1 *Assume that C is closed. If $P(x, A)$ has asymptotically bounded mean, then it satisfies uniform asymptotic tightness.*

This result follows because the expected value of a random variable provides a bound on its tail probabilities by Markov's inequality. By assumption, the expected value of the process is bounded uniformly for all initial values x as n becomes large. As a result, the bound on tail probabilities depends only on the common bound for the expected values K given in Definition 3, and not on the initial conditions. This is exactly the condition for uniform asymptotic tightness.[10]

[10] The assumption that C is closed is important for the result. To see why, let $C = \{x \in \mathbb{R} |\ x > 0\}$. Then $x_{t+1} = \min\left[x_t^2, 1\right]$ clearly has asymptotically bounded mean, but it does not satisfy asymptotic tightness, as all the mass escapes towards zero for this process.

Example. For concreteness, consider the following process:

$$x_{t+1} = A_{t+1}x_t + b_{t+1} \tag{4}$$

where $A_{t+1} \geq 0$ and $b_{t+1} \geq 0$ are independent i.i.d. random variables, $\mathrm{E}b < \infty$, and let $C = \{x \in \mathbb{R} |\ x \geq 0\}$. As we will see in Section 2, both economic applications in the paper lead to dynamics similar to (4).

For the process (4), we can bound $\mathrm{E}x_{t+1}$ when $\mathrm{E}A < 1$ by noting that repeated substitution implies $\mathrm{E}|x_{t+1}| < (\mathrm{E}A)^{t+1}\, x_0 + \mathrm{E}b/\,(1 - \mathrm{E}A)$. In particular, as $t \to \infty$, the expected value is bounded by a constant $K = 2\mathrm{E}b/\,(1 - \mathrm{E}A)$ independently of the initial value x_0. Thus, when $\mathrm{E}A < 1$ this process satisfies asymptotically bounded mean and thus also uniform asymptotic tightness.

As the example shows, asymptotically bounded mean is easy to verify in practice. However, it can be a strong requirement: in both economic applications considered below, for a set of parameters, asymptotic tightness obtains even though the mean of the processes is unbounded. To understand how this can happen, consider (4) for large values of x_t. For large x_t, the value of x_{t+1} is essentially determined by $A_{t+1}x_t$. Taking logs, we have approximately

$$\log x_{t+1} \approx \log A_{t+1} + \log x_t \tag{5}$$

a random walk with innovation $\log A_{t+1}$. This equation suggests that the process does not escape to infinity as long as the random walk (5) has negative drift, i.e., the innovation satisfies $\mathrm{E}\log A < 0$, a weaker condition than $\mathrm{E}A < 1$. Similarly, when $\mathrm{E}\log A > 0$ the drift is positive, and hence the process is likely to escape to infinity and fail asymptotic tightness.

I now develop a formal version of this argument that can be used in applications. The idea is to make use of an upper bound process y_{t+1} which is always greater than or equal to $\log x_{t+1}$, and for which uniform asymptotic tightness is easy to verify. Set y_{t+1} to follow a random walk with innovation $\log A_{t+1}$ when sufficiently far away from zero, and prevent it from getting close to zero using a lower bound which may be stochastic. The process y_{t+1} defined this way, which I call a random walk with a stochastic reflecting barrier, serves as an upper bound for $\log x_{t+1}$ as long as the barrier is high enough relative to b_{t+1}. The Proposition below shows that y_{t+1} is asymptotically tight when $\mathrm{E}\log A_{t+1} < 0$; hence so is $\log x_{t+1}$. A similar construction of a random walk with absorbing barrier for a lower bound can be used to show that when $\mathrm{E}\log A_{t+1} > 0$, the process $\log x_{t+1}$ does escape to infinity.

The following proposition states the technical results for random walks with barriers required for the above argument.

Proposition 2 *Let (u_t, v_t) be a sequence of i.i.d. random pairs, with $E|\nu_t| < \infty$ and $E|u_t| < \infty$.*

(i) [Reflecting barrier] If $Eu_{t+1} < 0$, the stochastic process

$$y_{t+1} = u_{t+1} + \max\left[y_t, \nu_{t+1}\right]$$

is monotone, Feller, and satisfies uniform asymptotic tightness

(ii) [Absorbing barrier] If $Eu_t > 0$, for any ν real the stochastic process

$$y_{t+1} = \begin{cases} y_t + u_{t+1} & \text{if } y_t > \nu \\ \nu_{t+1} & \text{if } y_t \leq \nu \end{cases}$$

does not satisfy uniform asymptotic tightness.

The process in (i) can be thought of as a random walk with a reflecting stochastic barrier. When y_t is large, y_{t+1} is most likely determined as $y_{t+1} = u_{t+1} + y_t$, as in a random walk. When y_t is small, it may fall below the stochastic barrier ν_{t+1}, in which case $y_{t+1} = u_{t+1} + \nu_{t+1}$ is reflected from the barrier ν_{t+1}. Monotonicity, continuity and mixing of this process follow from standard arguments. When the innovation $\mathrm{E}u < 0$, this process also satisfies asymptotic tightness. The intuition is simple: a high realization of y_{t+1} requires a sequence of high realizations of u_{t+1}, but this is unlikely if $\mathrm{E}u_{t+1} < 0$ because of the law of large numbers. This confirms the logic of the argument stated before the proposition.

The process in (ii) is a random walk with a stochastic absorbing barrier. Here, the absorbing barrier if it is ever hit, keeps the value of the process down. But if the barrier is avoided, then $\mathrm{E}u_{t+1} > 0$ implies that the process follows a random walk with positive drift and hence escapes to infinity.

2 Applications

2.1 Buffer Stock saving

Setup. I consider a setup similar to Carroll (1997, 2004) and Haliassos and Michaelides (2003). An infinitely lived agent has stochastic labor income and can invest in a menu of two financial securities: a risky stock and a safe bond. The agent faces borrowing and short sales constraints, so that the portfolio shares of his wealth are restricted to be between zero and one for both assets. The agent solves

$$\max_C E_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to the period budget constraint

$$C_t + B_t + S_t = X_t$$

where B_t and S_t are the dollar amounts invested in bonds and stocks, and X_t is cash on hand available in period t. The evolution of X_t is given by

$$X_{t+1} = S_t R_{t+1} + B_t R_f + Y_{t+1}$$

where R_{t+1} is the risky rate of return on stocks, R_f is the riskfree rate earned by bonds and Y_{t+1} is stochastic labor income. I assume that

$$S_t \geq 0$$
$$B_t \geq 0$$

so that the consumer is not able to borrow or short sell the stock.[11] Period utility has constant relative risk aversion $\rho > 1$:

$$u(C_t) = \frac{C_t^{1-\rho}}{1-\rho}.$$

Following Carroll (1992), I model labor income Y_{t+1} as

$$Y_{t+1} = P_{t+1} \cdot \epsilon_{t+1}$$
$$P_{t+1} = P_t \cdot GN_{t+1}.$$

Here P_t is the permanent component and ϵ_t is the transitory component of labor income. P_t follows an exponential random walk with mean growth rate G and permanent shocks N_t with $\mathrm{E}N_t = 1$. I assume that N_t and ϵ_{t+1} are non-negative, independent and i.i.d., that $\mathrm{E}\epsilon_{t+1} < \infty$ and that $\epsilon_{t+1} > 0$ with positive probability. I also assume that the gross stock return R_{t+1} is non-negative,

[11] If the lower bound of labor income is zero, the non-negativity of total savings will arise endogenously (see Carroll, 1997).

i.i.d., independent of labor income, $\mathrm{E}R_{t+1} \geq R_f$ is finite, and that the excess return $R_{t+1} - R_f$ takes on both non-negative and non-positive values with positive probability. The special case with no risky asset, which is the specification considered by Carroll (1997, 2004), can be obtained by assuming that $R_{t+1} = R_f$. I also assume that there is some uncertainty in the economy, so that N_{t+1}, ϵ_{t+1} and R_{t+1} are not all degenerate random variables. The above assumptions are generally satisfied in consumption and portfolio choice models, where R_{t+1} and N_{t+1} are often taken to be lognormally distributed and ϵ_{t+1} is assumed to be bounded.[12] The non-negativity of R_{t+1} captures the limited liability of stocks.

As Deaton (1991), Carroll (2004) and Haliassos and Michaelides (2003) show, the following "impatience" conditions are useful for ensuring the existence of a solution to the consumer's problem:

$$\beta R_f \ \mathrm{E}_t[(GN_{t+1})^{-\rho}] < 1 \tag{6}$$

and

$$\beta \ \mathrm{E}_t[R_{t+1}(GN_{t+1})^{-\rho}] < 1. \tag{7}$$

These impatience conditions ensure that a Bellman operator associated with the consumer's problem is a contraction mapping, leading to a well-defined value function and consumption function.[13]

Due to the homogeneity of the utility function, the consumer's problem can be rewritten in ratio form, where all variables are normalized by the level of permanent income P_t. Denoting the normalized variables by lowercase letters ($x_t = X_t/P_t$, $c_t = C_t/P_t$, etc.) the dynamic of cash on hand becomes

$$x_{t+1} = (R_{p,t+1}/GN_{t+1})(x_t - c_t) + \epsilon_{t+1} \tag{8}$$

where $R_{p,t+1}$ denotes the return on the household's portfolio between t and $t+1$ and $x_t - c_t = s_t + b_t$ by definition.

Results. The theoretical results of Section 1 allow for an essentially complete characterization of the parameters for which a stable invariant distribution exists. As is shown below, an invariant distribution exists when consumption growth in an auxiliary model with no labor income is less

[12] See for example Carroll (1997, 2004), Haliassos and Michaelides (2003), Deaton (1991) and Campbell and Viceira (2002).

[13] Carroll (2004) shows that an additional restriction, the "nonpathological patience condition" $(R_f\beta)^{1/\rho} < R_f$ is also needed to guarantee that a solution exists. This condition automatically holds in the current setup because $\rho > 1$.

than permanent income growth in the buffer stock model.

For the formal analysis, consider the version of the buffer stock model with no labor income ($G = 0$). This is our auxiliary model, and it is equivalent to the discrete time Merton consumption problem with the additional restriction that the consumer is not allowed to short stocks.[14] As is well known, in the solution of this maximization problem, consumption is proportional to current resources and the portfolio share of stocks is constant. Let b^* denote the optimal share of consumption out of current wealth X_t, so that $C_t = b^* X_t$, let α^* be the optimal stock share in the consumer's portfolio, and write $R^*_{p,t+1} = \alpha^* R_{t+1} + (1 - \alpha^*) R_f$. Note that while the consumption and portfolio are chosen dynamically, in the optimal policy, b^* and α^* are constant, and hence no time subscript is necessary. While in general there are no closed form expressions for b^* and α^* in the discrete time Merton consumption problem, accurate log-linear approximations are available, and hence these parameters may be treated as essentially known.[15]

The following is the main result of this section.

Proposition 3 *The processes x_t, c_t, s_t and b_t have stable invariant distributions if*

$$E\log\left[R^*_{p,t+1}(1-b^*)\right] < E\log\left[GN_{t+1}\right]. \tag{9}$$

Moreover, if

$$E\log\left[R^*_{p,t+1}(1-b^*)\right] > E\log\left[GN_{t+1}\right] \tag{10}$$

then x_t does not have an invariant distribution.

Except for the knife-edge case when (9) holds with equality, the proposition completely characterizes the circumstances when the buffer stock model has a stable invariant distribution. It is important to note that the statement of the Proposition is not "circular": the conditions are in terms of R^*_p and b^*, which are endogenous variables of the auxiliary model, but exogenous to the original buffer stock model. To understand the intuition for the result, note that the left hand side of (9) is the expected value of log consumption growth in the Merton economy with no labor income, while the right hand side is the expected log growth rate of the permanent component of

[14] This restriction will be automatically satisfied if R_{t+1} is not bounded away from zero.

[15] If R_{t+1} is lognormally distributed, then up to a log-linear approximation $\log(1-b^*) = \frac{\log\beta}{\rho} + \frac{r_f}{\rho} - r_f - \frac{\mu^2}{\sigma^2}\frac{\rho-1}{2\rho^2}$ and $\alpha^* = \mu/\rho\sigma^2$ where $r_f = \log R_f$, $r_{t+1} = \log R_{t+1}$, $\sigma^2 =$var$[r_{t+1}]$ and $\mu =$E$[\log R_{t+1}] - r_f + \sigma^2/2$ which is approximately equal to the equity premium E$R_{t+1} - R_f$. These formulas are valid as long as $\alpha^* < 1$, otherwise the continuous time Merton model does not obtain as a limit of the discrete time version.

income in the buffer stock model.[16] When the former is smaller, the consumption to income ratio is mean-reverting and hence a stable invariant distribution exits. Mean reversion obtains because when the consumption to income ratio is high, consumption behavior is similar to the auxiliary model with no labor income, and hence, by (9), consumption is expected to grow slower than income.

The Proposition is useful because the auxiliary model is much easier to solve than the original one. In the special case when there is no risky asset, consumption growth in the Merton model can be computed analytically (see Carroll, 2004): $\alpha^* = 0$ and $1 - b^* = (R_f \beta)^{1/\rho} / R_f$. In this case, (9) becomes

$$\log \left[(R_f \beta)^{1/\rho} \right] < \mathrm{E} \log \left[G N_{t+1} \right],$$

which follows from the impatience condition (6) by Jensen's inequality since the exponential function is convex. As a result, in Carroll's specification with no risky asset, an invariant distribution always exists under the standard impatience assumption.

At this point, it is useful to note that the impatience condition (6) used in the literature is a sufficient but not necessary condition for the model with no risky asset to have a solution. For example, in the special case with no transitory shocks and constant permanent income ($G = 1$), the model will have a solution even if $\beta R_f > 1$, which violates the standard impatience condition (6).[17] In this case, consumption grows at a constant positive rate and hence does not have an invariant distribution, which is consistent with the above result, since the key condition in the Proposition, inequality (9), does not hold.[18] Thus there do exist cases where the consumer's problem has a solution but there is no stable invariant distribution. However, the standard impatience condition (6) used in the literature is strong enough that it ensures both a solution and a stable invariant distribution.

In the case with a risky and a riskfree asset, it can be shown that the key condition (9) in the Proposition is in general *not* implied by the standard impatience conditions (6) and (7). As a result, there are parameters where the consumer's problem has a solution, but consumption and cash-on-hand do not have invariant distributions. This case parallels the example discussed in the

[16] The left hand side is log consumption growth because in the Merton economy, the consumption to wealth ratio is constant and hence consumption growth equals wealth growth, which is $R^*_{p,t+1} (1 - b)$ since a share $(1 - b)$ of current wealth is saved and invested.

[17] As discussed in footnote 13, the assumption that $\rho > 1$ implies the nonpathological patience condition $(R_f \beta)^{1/\rho} < R_f$ of Carrol (2004), and hence a solution exists.

[18] Strictly speaking, this example does not satisfy all modelling assumptions, as there is no uncertainty. This inconsistency can be resolved by introducing a small transitory shock to labor income.

previous paragraph: in the long run, consumption is expected to grow at a faster rate than income, and hence their ratio is not mean-reverting. When there is both a risky and a safe asset, this can occur even if model parameters are such that the sufficient conditions (6) and (7) are satisfied and thus the consumer's problem does have a solution.

The result of the Proposition has implications for the predictability of consumption growth. When a stable invariant distribution exists, the consumption to permanent income ratio c_t has mean-reverting dynamics. As a result, high values of $c_t = C_t/P_t$ must eventually be followed by either lower consumption growth or higher income growth to bring down c_t to its long term stationary distribution. Since the growth of permanent income is by construction i.i.d. and hence unpredictable, it must be that high values of C_t/P_t predict lower than average subsequent consumption growth. In particular, positive temporary shocks which raise c_t predict lower subsequent consumption growth, i.e., the immediate consumption response to the shock is "too large." For these parameters, then, the model yields predictable consumption growth and excess sensitivity of consumption to temporary shocks, key features of buffer stock behavior (Carroll, 1997) which are inconsistent with the standard permanent income hypothesis. In contrast, when no stable invariant distribution exists, the model eventually converges to the solution of the Merton consumption problem, and hence in the limit consumption is a random walk whose innovations are determined by the innovation in stock returns. For these parameters, the standard permanent income hypothesis obtains: in the long run consumption growth is unpredictable and does not display excess sensitivity to temporary shocks. These results show that the existence of a stable invariant distribution has powerful implications for consumption behavior.

The proof of Proposition 3 proceeds by verifying the conditions of Theorem 1 for the process x_{t+1}. Monotonicity—though not trivial to verify due to the presence of a risky investment opportunity—follows from standard arguments presented in the Appendix. Verifying mixing is straightforward. The key difficulty is to show that x_{t+1} is asymptotically uniformly tight. To see the logic for tightness, note that for x large, the consumer's problem is close to an unconstrained Merton consumption problem. As a result, optimal consumption and investment will be similar to the Merton case, and cash on hand follows the approximate dynamic

$$\widetilde{x}_{t+1} = R^*_{p,t+1} \left(1 - b^*\right) / G N_{t+1} \cdot \widetilde{x}_t$$

where I substituted the Merton consumption and investment policy in equation (8), and ignored the labor income term ϵ_{t+1} because it is small relative to cash on hand. Let $u_{t+1} = \log\left[R^*_{p,t+1}(1-b^*)/GN_{t+1}\right]$ and $\widetilde{y}_{t+1} = \log \widetilde{x}_{t+1}$, then

$$\widetilde{y}_{t+1} = u_{t+1} + \widetilde{y}_t$$

approximates the dynamic of log cash on hand for x large. Making use of Proposition 2, we can conclude that when $\mathrm{E}u_{t+1} < 0$ the process satisfies asymptotic tightness, but when $\mathrm{E}u_{t+1} > 0$ it does not. Finally, invariance is established by showing that an uppoer bound process closely related to $\widetilde{y}_{t+1}$ statisfies the conditions of the Theorem and hence a probabilistic upper bound for x_{t+1} exists; while the distribution of ϵ_{t+1} serves as a probabilistic lower bound.

2.2 Stochastic Growth

In this section I explore the one-sector stochastic growth model first developed by Brock and Mirman (1972). The existence and stability of an invariant distribution has been the subject of a number of papers. Building on an earlier literature which focuses on bounded shocks, Stachurski (2002), Kamihigashi (2007) and Zhang (2007) derive the existence and stability of an invariant distribution with unbounded shock and capital stock. These papers work either under the assumption that either the technology shocks enter the production function multiplicatively or that the shock distribution is absolutely continuous.

Using the tools developed in Section 1, I explore conditions under which a stable distribution exists even for unbounded shocks and capital stock, and an arbitrary production function. I show that when the production function is log-supermodular, existence of a stable distribution follows from Theorem 1 under weak conditions. The restriction to log-supermodularity implies that technology and capital are weakly complementary, which is satisfied for example by all CES functions with elasticity of substitution less than or equal to one, including the multiplicative formulation.

Setup. The consumer solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the feasibility constraint $c_t \leq x_t$, where x_t denotes resources available in period t. The utility function u is increasing and concave, with $\lim_{c\to 0} u'(c) = \infty$. The accumulation of resources is governed by

$$x_{t+1} = f(x_t - c_t, a_{t+1})$$

where f is a production function and a_{t+1} is an i.i.d. non-degenerate random disturbance term.

I assume that $f(k,a)$ is increasing in both arguments, continuously differentiable and concave in k for all a, and satisfies the Inada conditions: $\lim_{k\to 0} f'(k,a) = \infty$ and $\lim_{k\to\infty} f'(k,a) = 0$ for all $a > 0$. In addition, let $f(0,a) = 0$ for all a and $f(k,0) = 0$ for all k. The special case when $f(k,a) = af(k)$ is analyzed by Stokey, Lucas and Prescott (1989), Hopenhayn and Prescott (1992) and Stachurski (2002), among others.

Results. I now establish conditions under which a stable invariant distribution exists when the production function is log-supermodular.[19]

Proposition 4 *If $E|\log f(k,a)|$ exists for all k and $f(k,a)$ is log-supermodular, then the Brock-Mirman model has a stable invariant distribution.*

When $f(k,a) = af(k)$, log-supermodularity is immediate, and thus a stable invariant distribution exists as long as $|\log a|$ has finite mean. This is weaker than Stachurski's (2002) and Zhang's (2007) condition that $1/a$ has a finite expect value, and does not require, unlike Kamihigashi (2007), that the distribution of a be absolutely continuous.

The proof of Proposition 4 proceeds by verifying the conditions of Theorem 1 for the process x_{t+1}. Mixing and monotonicity follow from standard arguments. To establish asymptotic tightness, consider $\log x_{t+1}$. In the buffer-stock application we had $x_{t+1} \geq \epsilon_{t+1}$, which implied that x_{t+1} could never get stuck at zero. In the current application, if x_{t+1} ever hits zero, it will continue to remain there, because $f(0,a) = 0$. As a result, we need to establish asymptotic tightness "from below" as well as from above for $\log x_{t+1}$.

For simplicity, I illustrate the argument of the proof using the special case where $f(k,a) = af(k)$. Since $f(x)/x \to 0$ as $x \to \infty$ by the Inada conditions, there exists $\overline{x}$ such that for $x > \overline{x}$ we have $\log f(x) < \log x - \mathrm{E}\log a - 1$. Let $u_{t+1} = \log a_{t+1} - \mathrm{E}\log a - 1$, then we have $\mathrm{E}u_{t+1} < 0$, and for $x_t > \overline{x}$

$$\log x_{t+1} \leq u_{t+1} + \log x_t.$$

Thus for x large, the process of $\log x_{t+1}$ is dominated by a random walk with negative drift. For x small, the process can be bounded by a constant related to $\log f(\overline{x})$, and then Proposition 2 (i) shows that the process x is asymptotically tight "from above."

To obtain a lower bound, I build on a lemma presented in the Appendix, which extends Lemma 7.2 in Stachurski (2002). The lemma shows that for any a^* in the support of a, when x is small

[19] Recall that a function $f(k,a)$ is log-supermodular if for all $k' \geq k$ and $a' \geq a$ we have $\log f(k',a') - \log f(k,a') \geq \log f(k',a) - \log f(k,a)$.

enough, $a^* f(x - c(x)) \geq x$ holds. This result can be used to show that for x_{t+1} small, the dynamic of $\log x_{t+1}$ is bounded from below by a random walk with positive drift. For x large, $\log x$ can be bounded by a constant, and another application of Proposition 2 (i) shows that x_{t+1} satisfies asymptotic tightness "from below."

Finally, to prove that the stable distribution is invariant I show that both the upper and lower bound processes satisfy the Theorem and hence have invariant distributions which serve as probabilistic upper and lower bounds for x_{t+1}.

3 Conclusion

This paper developed simple sufficient and almost necessary conditions for the existence of a stable invariant distribution of an increasing Markov process defined on a potentially non-compact state space. The conditions were used to establish the existence of stable invariant distributions in models of buffer-stock saving and stochastic growth. I hope that these sufficient conditions will be useful in other applications as well.

Appendix: Proofs

Preliminaries. I begin with some definitions. For any probability measure λ and bounded, measurable function f, define the inner product

$$\langle f, \lambda \rangle = \int_{\mathbb{R}^N} f(x) d\lambda(x).$$

For a real-valued, bounded, measurable functions f defined on C, let

$$Tf(z) = \int_{\mathbb{R}^N} f(x) P(z, dx)$$

the conditional expectation of f evaluated at the next realization of the process given that the current state is z. One can think of T as an operator defined on a function space; with this interpretation, T^* is the adjoint of T in the sense that $\langle f, T^*\lambda \rangle = \langle Tf, \lambda \rangle$ for all f and λ (see Stokey, Lucas and Prescott, 1989, p. 218, Corollary).

Consider a process that satisfies uniform asymptotic tightness. For each C_δ there exist $\overline{k}_\delta$ and $\underline{k}_\delta$ such that $C_\delta \subseteq \left[\underline{k}_\delta, \overline{k}_\delta\right]$ and $c \in \left[\underline{k}_\delta, \overline{k}_\delta\right]$. Moreover, since C is an interval set, $\overline{k}_\delta$ and $\underline{k}_\delta$ can be chosen from C, which ensures that $\left[\underline{k}_\delta, \overline{k}_\delta\right] \subseteq C$ also holds. Then, for any $x \in C$ there exists n large enough that $\Pr\left(\left|T^{*n}\delta_x\right| \in \left[\underline{k}_\delta, \overline{k}_\delta\right]\right) > 1 - \delta$ by definition.

Lemma 1 *If an increasing Markov process satisfies weak mixing and uniform asymptotic tightness, then there exists $\varepsilon > 0$ with the property that for all $\delta > 0$ there is $M > 0$ such that for any f non-decreasing, bounded function and $m \geq M$*

$$T^m f(\bar{k}_\delta) \leq \varepsilon f(c) + (1-\varepsilon) f(\bar{k}_\delta) + 2\delta \|f\|$$

and

$$T^m f(\underline{k}_\delta) \geq \varepsilon f(c) + (1-\varepsilon) f(\underline{k}_\delta) - 2\delta \|f\|.$$

Proof. By (3), there exists $\varepsilon_1 > 0$ such that $P^{\bar{S}}\left(\overline{k}_{1/2}, (-\infty, c)\right) > 2\varepsilon_1$. Since T^* is an increasing operator, it follows that for all $z \leq \overline{k}_{1/2}$ we have $P^{\bar{S}}\left(z, (-\infty, c)\right) > 2\varepsilon_1$. Now consider the n-step transition probability distribution starting from some $x \in C$.

$$P^n(x, .) = T^{*n}\delta_x = T^{*\bar{S}}\ T^{*(n-\bar{S})}\ \delta_x.$$

For all n large enough, $T^{*(n-\bar{S})}\ \delta_x$ assigns at least probability 1/2 to $\left[\underline{k}_{1/2}, \overline{k}_{1/2}\right]$. Moreover, for all $z \in \left[\underline{k}_{1/2}, \overline{k}_{1/2}\right]$, $T^{*\bar{S}}$ assigns at least probability $2\varepsilon_1$ to $(-\infty, c)$. Combining these observations shows that for any $x \in C$, $\lim_{n\to\infty} \inf P^n(x, (-\infty, c]) > \varepsilon_1$. A similar argument shows the existence of $\varepsilon_2 > 0$ such that for all $x \in C$, $\lim_{n\to\infty} \inf P^n(x, [c, \infty)) > \varepsilon_2$. Let $\varepsilon = \min[\varepsilon_1, \varepsilon_2]$.

Fix any $\delta > 0$. The above argument shows that for all m large enough

$$P^m(\bar{k}_\delta, (-\infty, c]) > \varepsilon \tag{11}$$

and analogously

$$P^m(\underline{k}_\delta, [c, \infty)) > \varepsilon. \tag{12}$$

Moreover, for all m large

$$P^m(\bar{k}_\delta, (-\infty, \bar{k}_\delta]) > 1 - \delta \tag{13}$$

and

$$P^m(\underline{k}_\delta, [\underline{k}_\delta, \infty)) > 1 - \delta \tag{14}$$

by definition. Set M such that (11) through (14) are satisfied for all $m \geq M$, and let f be a non-decreasing bounded function. Then

$$\begin{aligned} T^m f(\bar{k}_\delta) &= \\ &= \int_{(-\infty,c]} f(z) P^m(\bar{k}_\delta, dz) + \int_{(-\infty,\bar{k}_\delta]\backslash(-\infty,c]} f(z) P^m(\bar{k}_\delta, dz) + \int_{\mathbb{R}^N\backslash(-\infty,\bar{k}_\delta]} f(z) P^m(\bar{k}_\delta, dz) \\ &\leq P^m(\bar{k}_\delta, (-\infty, c]) f(c) + P^m(\bar{k}_\delta, (-\infty, \bar{k}_\delta]\backslash(-\infty, c]) f(\bar{k}_\delta) + P^m(\bar{k}_\delta, \mathbb{R}^N\backslash(-\infty, \bar{k}_\delta]) \|f\| \leq \\ &\leq \varepsilon f(c) + (1-\varepsilon) f(\bar{k}_\delta) + 2\delta \|f\| \end{aligned}$$

using (11) and (13). This proves the first half of the lemma. The second half can be established by a similar argument using (12) and (14). QED

Proof of Theorem 1. Begin with part 1. Let x_1, $x_2 \in C$. I will show that if $j_1, j_2 \to \infty$, perhaps along subsequences, then $T^{*j_1}\delta_{x_1}$ and $T^{*j_2}\delta_{x_2}$ converge to the same limit distribution. Fix any $\delta > 0$, this defines $\overline{k}_\delta$ and $\underline{k}_\delta$ as above. For ease of notation, I will no longer indicate the dependence of $\bar{k}$ and $\underline{k}$ on δ. There exists n_0 (which depends on δ) such that for all $n \geq n_0$ we have $P^n(x_1, [\underline{k}, \bar{k}]) > 1 - \delta$ and $P^n(x_2, [\underline{k}, \bar{k}]) > 1 - \delta$. Let g be a non-decreasing, continuous, bounded function and $n \geq n_0$, then for all $j_1 > n$

$$\begin{aligned}
\left\langle T^{j_1} g, \delta_{x_1} \right\rangle &= \int T^{j_1} g(z)\ P\left(x_1, \mathrm{d}z\right) = \int T^{j_1-n} g(z)\ P^n\left(x_1, \mathrm{d}z\right) = \\
&= \int_{(-\infty,\bar{k}]} T^{j_1-n} g(z)\ P^n\left(x_1, \mathrm{d}z\right) + \int_{\mathbb{R}^N \setminus (-\infty,\bar{k}]} T^{j_1-n} g(z)\ P^n\left(x_1, \mathrm{d}z\right) \leq \\
&\leq (1-\delta) T^{j_1-n} g(\bar{k}) + \delta \cdot \|g\| \leq \\
&\leq T^{j_1-n} g(\bar{k}) + 2\delta \|g\| .
\end{aligned}$$

A similar argument shows that $\left\langle T^{j_1} g, \delta_{x_1} \right\rangle > T^{j_1-n} g(\underline{k}) - 2\delta \|g\|$ and similar inequalities hold for x_2 as well. Combining these shows that for all $j \leq \min [j_1, j_2] - n_0$

$$T^j g(\underline{k}) - 2\delta \|g\| \leq \ \left\langle T^{j_1} g, \delta_{x_1} \right\rangle \ \leq T^j g(\bar{k}) + 2\delta \|g\| \tag{15}$$

and

$$T^j g(\underline{k}) - 2\delta \|g\| \leq \ \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \ \leq T^j g(\bar{k}) + 2\delta \|g\| . \tag{16}$$

Write $\min [j_1, j_2] - n_0 = mM + n_1$ where $0 \leq n_1 < M$ and M is as defined in Lemma 1. Then (15) and (16) imply

$$\left| \left\langle T^{j_1} g, \delta_{x_1} \right\rangle - \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \right| \leq T^{Mm} g(\bar{k}) - T^{Mm} g(\underline{k}) + 4\delta \|g\| . \tag{17}$$

Now note that $f = T^{M(m-1)} g$ is a bounded, non-decreasing function, and thus satisfies the conditions of Lemma 1. Since $\|f\| \leq \|g\|$, the Lemma, combined with (17), yields

$$\left| \left\langle T^{j_1} g, \delta_{x_1} \right\rangle - \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \right| \leq (1-\varepsilon) \left[T^{M(m-1)} g(\bar{k}) - T^{M(m-1)} g(\underline{k}) \right] + 8\delta \|g\| . \tag{18}$$

We can iterate the right hand side in (18) by applying Lemma 1 repeatedly, first for $f = T^{M(m-2)} g$, then $f = T^{M(m-3)} g$, and so on. This yields

$$\left| \left\langle T^{j_1} g, \delta_{x_1} \right\rangle - \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \right| \leq (1-\varepsilon)^m \left[g(\bar{k}) - g(\underline{k}) \right] + 4\delta \|g\| + 4\delta \|g\| \left\{ 1 + (1-\varepsilon) + ... + (1-\varepsilon)^{m-1} \right\}$$

and summing the terms on the right hand side implies

$$\left| \left\langle T^{j_1} g, \delta_{x_1} \right\rangle - \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \right| \leq \|g\| \left\{ 2(1-\varepsilon)^m + 4\delta + \frac{4\delta}{\varepsilon} \right\} . \tag{19}$$

Now consider any subsequences of j_1 and j_2. For any δ, as j_1 and j_2 grow without bound along their subsequences, m will also grow without bound, because $m \geq (\min [j_1, j_2] - n_0) / M - 1$. As a result, the right hand side in (19) can be made smaller than $4 \|g\| \delta (1 + 1/\varepsilon)$ for any δ. Since ε is fixed, as $\delta \to 0$ this number will be arbitrarily small. As a result, any subsequence of j_1 and j_2 has a subsequence along which $\left| \left\langle T^{j_1} g, \delta_{x_1} \right\rangle - \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \right|$ converges to zero. But then $\lim_{j_1, j_2 \to \infty} \left| \left\langle T^{j_1} g, \delta_{x_1} \right\rangle - \left\langle T^{j_2} g, \delta_{x_2} \right\rangle \right| = 0$ must also hold.

To proceed, note that the sequence of measures $T^{*j_1} \delta_{x_1}$ is tight by assumption. Therefore, by Prokhorov's theorem, there exists a convergent subsequence and a limit distribution μ. Suppose now that the entire sequence $T^{*j_1} \delta_{x_1}$ does not converge to μ; then there exists a second subsequence with a different limit μ'. But the above argument with $x_1 = x_2$ then implies that

$$\int_{\mathbb{R}^N} g(x) d\mu(x) = \int_{\mathbb{R}^N} g(x) d\mu'(x) \tag{20}$$

for all g monotone, bounded and continuous functions. It is a well-known fact that if (20) holds for all continuous and bounded functions, then $\mu = \mu'$. Here I argue that even if (20) holds only for monotone bounded and continuous functions, $\mu = \mu'$ still follows. To see the logic, note that for each set of the form $[x, \infty)$, one can construct a decreasing sequence of monotone functions g_l such that $0 \leq g_l \leq 1$, $g_l(y) = 1$ if $y \in [x, \infty)$ and $g_l(y) = 0$ if $y \notin [x - 1/l, \infty)$. Then $\lim_{l\to\infty} \int g_l d\mu = \mu([x, \infty))$ and similarly for μ', which implies that $\mu([x, \infty)) = \mu'([x, \infty))$. But then $\mu([x, y)) = \mu'([x, y))$ must hold for all x and y, because all sets of the form $[x, y)$ can be obtained from sets of the form $[x, \infty)$ using the operations of disjoint union and set subtraction. Since half-open interval sets generate the entire Borel sigma-algebra, it follows that $\mu = \mu'$.

It follows that any subsequence of $T^{*j_1}\delta_{x_1}$ converges to μ, i.e., that the entire sequence converges. Thus $T^{*j_1}\delta_{x_1}$ is convergent for all x_1; the above argument for $x_1 \neq x_2$ now implies that all these sequences converge to the same limit distribution μ.

The above result implies that the process converges to the same limit distribution μ when started from any initial state x. For a non-degenerate initial distribution λ with compact support, we can find x_1 and x_2 such that $\delta_{x_1} \leq \lambda \leq \delta_{x_2}$. Since the process started from x_1 and x_2 converges to the same limit μ, so does the process started from λ. Finally, for an arbitrary initial distribution λ, let f be a bounded and continuous function, $\varepsilon > 0$, and set $x_1 < x_2$ such that $\lambda([x_1, x_2]) > 1 - \varepsilon$. Write $\lambda = \alpha\lambda_a + (1 - \alpha)\lambda_b$ where $\lambda_a([x_1, x_2]) = 1$, $\lambda_b([x_1, x_2]) = 0$, and, by design, $\alpha > 1 - \varepsilon$. Now, for n large

$$\begin{aligned}
|\langle f, T^{*n}\lambda\rangle - \langle f, \mu\rangle| &= \left| \int_C T^n f(x) d\lambda(x) - \int_C f(x) d\mu(x) \right| \\
&\leq \alpha \left| \int_C T^n f(x) d\lambda_a(x) - \int_C f(x) d\mu(x) \right| + (1 - \alpha) \left| \int_C T^n f(x) d\lambda_b(x) - \int_C f(x) d\mu(x) \right| \\
&\leq \alpha\varepsilon + 2(1 - \alpha)|f|
\end{aligned}$$

where we used that $T^{*n}\lambda_a$ converges to μ. The last term can be made arbitrarily small by setting ε small, showing that $T^{*n}\lambda$ converges to μ.

Continue with part 2. Suppose P is Feller, and let μ be the stable distribution. Stability of μ implies that for any g bounded and continuous, $|\langle f, T^{*n}\mu\rangle - \langle f, \mu\rangle|$ converges to zero in n. The Feller property implies that for any f bounded and continuous, Tf is also continuous. Let $g = Tf$, then $\left|\left\langle f, T^{*n+1}\mu\right\rangle - \langle f, T^*\mu\rangle\right|$ also converges to zero, implying that $\mu = \lim T^{*n}\mu = T^*\mu$.

Now suppose P has probabilistic upper and lower bounds. Then the Knaster-Tarski fixed-point theorem implies, the same way as in Hopenhayn and Prescott (1992), that an invariant distribution λ exists. Stability implies that $T^{*n}\lambda = \lambda$ converges to μ, hence $\lambda = \mu$ is both stable and invariant.

Finally, part 3. I begin by showing that uniform asymptotic tightness is necessary for the existence of a stable distribution. Let μ be the stable distribution, let U_δ be a bounded open set with $\mu(U_\delta) > 1 - \delta$, and let C_δ be the closure of U_δ in the Euclidean topology. Since U_δ is bounded, C_δ is compact. Moreover, since $P^n(x, .) \Longrightarrow \mu$, we have $\lim_{n\to\infty} \inf P^n(x, U_\delta) \geq \mu(U_\delta) > 1 - \delta$ where the first inequality follows because U_δ is open. But this implies that $\lim_{n\to\infty} \inf P^n(x, C_\delta) > 1 - \delta$ which is the definition of uniform asymptotic tightness.

Next I show that if μ is non-singular and stable, then weak mixing holds. Below I argue that if μ is non-singular then there exists c such that $\mu(\{x < c\}) > 0$ and $\mu(\{x > c\}) > 0$. If such a c exists, then the fact the process converges to μ from any initial condition immediately implies weak mixing with respect to c.

To see why a c with the above property exists, write $\mu = \mu_1 + \mu_2$ where μ_1 is absolutely continuous with respect to the Lebesgue measure, and $\mu_1(C) > 0$ by assumption. The claim is immediate for $N = 1$. Suppose that $N = 2$ and consider a unit square $[x, x+1]$ in the plane such that $\mu_1([x, x+1]) > 0$. If $\mu_1([x, x+1/2]) > 0$ and $\mu_1([x+1/2, x+1]) > 0$ both hold, then we are done, as $x + 1/2$ can serve as c. (Note, the boundaries of these interval sets can be ignored as μ_1 is absolutely continuous). Otherwise, the Lebesgue measure of the support of μ_1 in $[x, x+1]$ must be at most $3/4$, since either $\mu_1([x, x+1/2]) = 0$ or $\mu_1([x+1/2, x+1]) = 0$ must hold. Now repeat the above argument for the three remaining smaller squares in $[x, x+1]$. For each of these three squares, either their midpoint serves as a valid c, or μ_1 must be concentrated on a subset with Lebesgue measure at most $3/4$ of their size. As a result, if none of the midpoints serves as a valid c, it must be that μ_1 is concentrated on a set of measure $(3/4)^2$. Repeating the argument by splitting the squares into four squares indefinitely shows that if a c with the desired property does not exist, then μ_1 must be concentrated on a set of zero Lebesgue measure, since $(3/4)^j \to 0$ as $j \to \infty$. This contradicts the assumption that μ_1 is absolutely continuous and has positive measure on $[x, x+1]$. For higher dimensions N, a similar argument works replacing $3/4$ by $\left(2^N - 1\right)/2^N$.

Proof of Proposition 1. Let $K_\delta = \left\{x \in \mathbb{R}^N \middle|\ |x| \leq 2K/\delta\right\}$, then $P^n(x, K_\delta) > 1 - \delta$ for n large by the Markov inequality. Let $C_\delta = C \cap K_\delta$. Since C is closed and K_δ is compact, C_δ is compact and $P^n(x, C_\delta) > 1 - \delta$, as desired.

Proof of Proposition 2. (i) Monotonicity is immediate. The Feller property follows because addition and the max operator are continuous. To show tightness, begin by exploring the properties of a different process. Let (w_1, γ_1), (w_2, γ_2),... be independent random variables, all distributed according to the distribution of (u_1, ν_1). Consider

$$S_n = \sum_{i=1}^{n} w_i + \gamma_n.$$

By the strong law of large numbers, $S_n/n \to \mathrm{E}w_1$ with probability one. The logic of the proof is to bound the tail probability of S_n, and use this bound to establish asymptotic tightness for the process y_{t+1}. Fix some $u > 0$, and consider

$$\Pr[\exists n\colon\ S_n > u] \leq \Pr[\exists n < k\colon\ S_n > u] + \Pr[\exists n \geq k\colon\ S_n > u]. \tag{21}$$

We can rewrite the second term as

$$\begin{aligned}\Pr[\exists n \geq k\colon\ S_n > u] &= \Pr\left[\exists n \geq k\colon\ \frac{S_n - n\mathrm{E}w_1}{n} > \frac{u - n\mathrm{E}w_1}{n}\right] \leq \\ &\leq \Pr\left[\exists n \geq k\colon\ \frac{S_n - n\mathrm{E}w_1}{n} > -\mathrm{E}w_1\right].\end{aligned}$$

Recall that $\mathrm{E}w_1 < 0$. Since $(S_n - n\mathrm{E}w_1)/n \to 0$ with probability one, the term on the right hand side goes to zero as $k \to \infty$, independently of u. In addition, for any fixed k, the term $\Pr[\exists n < k\colon\ S_n > u]$ can be made arbitrarily small for u large enough. Combining these observations with (21) implies that $\Pr[\exists n\colon\ S_n > u] \to 0$ as $u \to \infty$.

Now consider the process y_{t+1} of the Proposition. Our goal is to estimate $\Pr[y_{t+1} > u]$. First note that

$$\Pr[y_{t+1} > u] \leq \Pr[\forall s \leq t:\ y_s > u_s + \nu_s] + \Pr\left[\exists s \leq t+1:\ y_s = u_s + \nu_s \text{ and } \nu_s + \sum_{j=s}^{t+1} u_j > u\right]. \tag{22}$$

I begin by bounding the second term in this expression. Let $w_i = u_{t+2-i}$ and let $\gamma_i = \nu_{t+2-i}$ for $i = 1, 2,...,t+1$, and define (w_i, γ_i) to be i.i.d. with distribution (u_1, ν_1) for $i > t+1$. Then (w_n, γ_n) for $n = 1, 2,.3,...$ are independent random variables, all distributed according to the distribution of (u_1, ν_1). Since these variables satisfy the assumptions made above for (w_n, γ_n), it follows that $\Pr[\exists n\colon S_n > u] \to 0$ as $u \to \infty$ also holds. But as long as $n \leq t+1$, we have $S_n = \sum_{i=1}^n w_i + \gamma_n = \nu_s + \sum_{j=s}^{t+1} u_j$ where $s = t+2-n$. Therefore we can bound the second term in (22) using S_n as

$$\Pr\left[\exists s \leq t+1:\ y_s = u_s + \nu_s \text{ and } \nu_s + \sum_{j=s}^{t+1} u_j > u\right] \leq \Pr[\exists n\colon S_n > u].$$

As we have seen, as $u \to \infty$ the bound on the right goes to zero. For any $\delta > 0$, set u to be large enough so that $\Pr[\exists n\colon S_n > u] < \delta/2$. Next I turn to bound the first term in (22). This is easy, because $\Pr[\forall s \leq t:\ y_s > u_s + \nu_s] \to 0$ as $t \to \infty$ for any y_0. Thus, for any given y_0, we can pick t_0 such that for all $t \geq t_0$, we have $\Pr[\forall s \leq t:\ y_s > u_s + \nu_s] < \delta/2$. Combining these inequalities implies that when y is started from y_0, $\Pr[y_{t+1} > u] < \delta$ for all $t > t_0$. Letting $C_\delta = [0, u]$, this means that $\lim_{t\to\infty} \inf \Pr[y_{t+1} \in C_\delta] > 1 - \delta$ for any initial value y_0, which establishes uniform asymptotic tightness of y_{t+1}.

(ii) A similar logic based on the strong law of large numbers can be used to show that as $y_0 \to \infty$, $\Pr\left[\forall t:\ y_0 + \sum_{s=0}^t u_s > \nu\right] \to 1$. Moreover, again using the strong law of large numbers, one can show that $\Pr\left[\forall t > t_0:\ y_0 + \sum_{s=0}^t u_s > u\right] \to 1$ as $t_0 \to \infty$ for any given u. Now suppose that y satisfies asymptotic tightness. Then for some $\delta > 0$ there must exist u such that $\lim_{t\to\infty} \inf \Pr[y_t < u] > 1 - \delta$. Note that

$$\Pr[y_t < u] \leq \Pr\left[\exists t':\ y_0 + \sum_{s=0}^{t'} u_s \leq \nu\right] + \Pr\left[\ y_0 + \sum_{s=0}^{t} u_s \leq u\right]. \tag{23}$$

Pick y_0 so that $\Pr\left[\forall t:\ y_0 + \sum_{s=0}^t u_s > \nu\right] > 1 - \delta/2$ and t_0 so that $\Pr\left[\forall t > t_0:\ y_0 + \sum_{s=0}^t u_s > u\right] > 1 - \delta/2$. Consider the process y_{t+1} started from y_0: by the above inequalities, for all $t > t_0$ the right hand side of (23) is less than δ, which contradicts $\lim_{t\to\infty} \inf \Pr[y_t < u] > 1 - \delta$ if $\delta < 1/2$. This shows that y_t does not satisfy asymptotic tightness.

Proof of Proposition 3. Start with the case where equation (9) holds. We need to verify the conditions of Theorem 1 for the process x_{t+1}. Begin with uniform asymptotic tightness. The approach I take is to show that asymptotic tightness for x_{t+1} follows from asymptotic tightness for a related process where the condition can be verified using Proposition 2.

Note that for given P_t, as $X_t \to \infty$, the optimal consumption and investment policy of the buffer stock model converges to that of the Merton consumption problem by the theorem of the maximum, since labor income constitutes a smaller and smaller share of total wealth. As a result, for every $\varepsilon > 0$ there exists $\overline{x}$ such that for $x > \overline{x}$ we have $|c(x)/x - b^*| < \varepsilon$ and $|\alpha(x) - \alpha^*| < \varepsilon$.

Choose $\varepsilon, \delta > 0$ small enough such that for

$$u_{t+1} = \log\left[\left(\alpha^* + \varepsilon\right) R_{t+1} + \left(1 - \alpha^* + \varepsilon\right) R_f\right] + \log\left[1 - b^* + \varepsilon\right] - \log\left[GN_{t+1}\right] + \log\left(1 + \delta\right)$$

we have $\mathrm{E}u_{t+1} < 0$. Such ε and δ exist by (9) and by the monotone convergence theorem.

Let $\nu_{t+1} = \log \max\left[\left(1 + 1/\delta\right) \epsilon_t, \overline{x}\right]$and consider the process y_t defined as

$$y_{t+1} = u_{t+1} + \max\left[y_t, \nu_{t+1}\right].$$

Since $\mathrm{E}\epsilon_{t+1} < \infty$, it follows that $\mathrm{E}\nu_{t+1} < \infty$, and $\mathrm{E}u_{t+1} < 0$ by assumption. Moreover, by truncating the distribution of u at a low negative value if necessary, we can ensure that $\mathrm{E}|u| < \infty$ since $\mathrm{E}R_{t+1} < \infty$ and $\mathrm{E}_t[R_{t+1}(GN_{t+1})^{-\rho}] < \infty$ by assumption. Proposition 2 (i) then implies that y_{t+1} is monotone, Feller, and satisfies uniform asymptotic tightness. It is easy to see that it also satisfies weak mixing. It follows that the same conditions also hold for $\exp y_{t+1}$, which therefore has a unique stable invariant distribution.

Now consider the non-negative Markov process $z_{t+1} = x_{t+1} - \epsilon_{t+1}$. The dynamics of $\exp y_{t+1}$ dominates that of z_{t+1} in the sense of first order stochastic dominance. To see why, note that for any initial value $z_0 = x_0 - \epsilon_0$, the following first-order stochastic dominance relation holds:

$$\log z_1 = \log\left[(R_p/GN_{t+1})(x_0 - c_0)\right] \preceq u_{t+1} - \log\left(1 + \delta\right) + \log \max\left[x_0, \overline{x}\right]$$

because x_{t+1} is an increasing process. Moreover, $x_0 = z_0 + \epsilon_0 \leq \max\left[\left(1 + \delta\right) z_0, \left(1 + 1/\delta\right) \epsilon_0\right]$ and hence $\log \max\left[x_0, \overline{x}\right] \leq \log\left(1 + \delta\right) + \max\left[\log z_0, \nu_1\right]$. This shows that the distribution of $\log z_1$ will be dominated by the distribution of y_1, and hence that the transition operator of $\exp y$ dominates that of z. Since $\exp y$ satisfies asymptotic tightness and dominates z which is non-negative, it follows that z also satisfies asymptotic tightness. Finally, $x_{t+1} = z_{t+1} + \epsilon_{t+1}$ is the sum of two non-negative asymptotically tight processes and hence asymptotically tight, as desired.

I now turn to establish monotonicity. Note that for $x_t' \geq x_t$, the optimal policies satisfy $s_t' \geq s_t$ and $b_t' \geq b_t$ because the continuation value function of the consumer exhibits decreasing absolute risk aversion (Carroll and Kimball, 1996), a property which is preserved in the presence of independent background risk (Gollier 2001, p. 116). As a result, by equation (8) the distribution of x_{t+1}' dominates that of x_{t+1} which shows that x is increasing.

Next consider mixing. Let $\underline{\epsilon}$ denote the infimum and $\overline{\epsilon}$ the supremum of the support of ϵ, and $\overline{N}$ the supremum of the support of N_{t+1}. If $\overline{N} = \infty$ then strong mixing follows immediately for any c such that $\underline{\epsilon} \leq c \leq \overline{\epsilon}$ because for any x_0, with a high realization of N_{t+1}, the process can, in just one step, reach a value arbitrarily close to $\underline{\epsilon}$; and similarly, for any x_0 in one step the process can reach a value greater than $\overline{\epsilon}$ minus any small number.

If $\overline{N} < \infty$, then consider the dynamic defined by

$$\widehat{x}_{t+1} = (R_f/G\overline{N})(\widehat{x}_t - c(\widehat{x}_t)) + \underline{\epsilon}. \tag{24}$$

I now show that the slope of the right hand side in $\widehat{x}$ is below one and bounded away from one. Kimball and Carroll (1996) show that the consumption function in this model is concave, so that $c'(\widehat{x})$ is decreasing. Moreover, $\widehat{x} - c(\widehat{x}) \leq (1 - b) \cdot \widehat{x}$ for all $\widehat{x}$ because the presence of future labor income cannot reduce current consumption. This implies that $1 - c'(\widehat{x}) \leq (1 - b) \cdot \widehat{x}$, because $1 - c'(\widehat{x})$ is increasing. As a result, the slope of the right hand side of (24) as a function of $\widehat{x}_t$ is less than $R_f\left(1 - b\right)/G\overline{N}$ which must be less than one by (9) This verifies that the slope of the right hand side is below 1.

It follows that a fixed point $\widehat{x}^*$ of the dynamics (24) uniquely exits, and the dynamics of $\widehat{x}$

converges to $\widehat{x}^*$ monotonically for any x_0. Because $R_{t+1} \leq R_f$ with positive probability, for any x_0 the dynamic of x will have realizations arbitrarily close to, or below, the dynamic of $\widehat{x}$. As a result, for any x_0, the process x ends up with positive probability below $\widehat{x}^* + \varepsilon$ for any $\varepsilon > 0$. Similarly, as $R_{t+1} \geq R_f$ with positive probability, the dynamics of x will have realizations above $\widehat{x}$ with positive probability, and hence ends up with positive probability strictly above $\widehat{x}^*$, which shows strong mixing.

We have verified that x_{t+1} has a unique stable distribution. To also show that this distribution is invariant, it suffices to note that since $\epsilon_{t+1} \leq x_{t+1}$ and $x_{t+1} - \epsilon_{t+1} \leq \exp y_{t+1}$, and because ϵ_{t+1} is independent of y_{t+1} and i.i.d. non-negative, the invariant distributions of ϵ_{t+1} and of $\exp y_{t+1} + \epsilon_{t+1}$ serve as probabilistic lower and upper bounds.

To show that c_t, s_t and b_t all have unique invariant distributions, note that all of these variables are continuous functions of x_t, and the continuous function of a weakly convergent random sequence is itself weakly convergent.

Finally, I show that when (10) holds, x does not satisfy uniform asymptotic tightness. We proceed using Lemma 2 (ii). Pick ε small enough that

$$u_{t+1} = \log\left[\left(\alpha^* - \varepsilon\right) R_{t+1} + \left(1 - \alpha^* - \varepsilon\right) R_f\right] + \log\left[1 - b^* - \varepsilon\right] - \log\left[G N_{t+1}\right]$$

satisfies $\mathrm{E}u_{t+1} > 0$, let $\nu_{t+1} = \log \epsilon_{t+1}$ and $\nu = \log \overline{x}$, and define

$$y_{t+1} = \begin{cases} y_t + u_{t+1} & \text{if } y_t > \nu \\ \nu_{t+1} & \text{if } y_t \leq \nu. \end{cases}$$

I argue that the dynamics of $z_{t+1} = \log x_{t+1}$ dominates that of y_{t+1}. This is because for $z_0 > \nu$ we have

$$z_1 = \log\left[(R_p / G N_{t+1})(x_0 - c_0) + \epsilon_{t+1}\right] \geq u_{t+1} + z_0$$

and for $z_0 \leq \nu$ we have $z_1 \geq \nu_1 = \log \epsilon_1$ by definition. Since $\mathrm{E}u_{t+1} > 0$, Lemma 2 (ii) implies that y_{t+1} does not satisfy uniform asymptotic tightness; but then neither does z_{t+1} and hence neither does x_{t+1}.

Proof of Proposition 4. The proof that the process satisfies asymptotic tightness from below requires the following lemma, which generalizes Lemma 7.2 in Stachurski (2002).

Lemma 2 *For each a^* that satisfies* $\Pr\left(a \leq a^*\right) > 0$*, there exists $\overline{x}$ such that for all $x \leq \overline{x}$ we have $f\left(x - c(x), a^*\right) \geq x$.*

Proof. Let $V\left(x\right)$ be the representative consumer's value function. The Euler equation together with the envelope condition implies that

$$\begin{aligned} V'\left(x\right) &= \beta \int_0^\infty V'\left(f\left(x - c(x), a\right)\right) \cdot f'\left(x - c(x), a\right) \mathrm{d}F\left(a\right) \\ &\geq \beta \int_0^{a^*} V'\left(f\left(x - c(x), a^*\right)\right) \cdot f'\left(x - c(x), a\right) \mathrm{d}F\left(a\right) \\ &\geq V'\left(f\left(x - c(x), a^*\right)\right) \cdot \beta \int_0^{a^*} f'\left(x - c(x), a\right) \mathrm{d}F\left(a\right). \end{aligned}$$

Since $f'(y,a) \to \infty$ as $y \to 0$ monotonically and $x - c(x)$ is monotone in x, the integral on the right hand side can be made arbitrarily large for x small. As a result, there exists $\overline{x}$ such that for $x \leq \overline{x}$

$$V'(x) > V'(f(x - c(x), a^*))$$

which implies that $f(x - c(x), a^*) \geq x$ for all $x \leq \overline{x}$. QED.

Now fix some $x^* > 0$, and for a fixed a^*, let $u_{t+1} = \inf_{x<x^*} \log[f(x,a)] - \log[f(x,a^*)]$. By log-supermodularity, it is easy to see that

$$u_{t+1} = \begin{cases} \log[f(0,a)/f(0,a^*)] & \text{if } a > a^* \\ \log[f(x^*,a)/f(x^*,a^*)] & \text{if } a \leq a^* \end{cases}$$

where I used the notation that $\lim_{x\to 0} \inf \log[f(x,a)/f(x,a^*)] = \log[f(0,a)/f(0,a^*)]$. Note that this term is by definition non-negative. By the assumption that $\log[f(x^*,a)]$ is integrable, there exists a^* small enough such that $\mathrm{E}u_{t+1} > 0$ and $\Pr(a \leq a^*) > 0$ still holds. Fix a^* at such a value.

By Proposition 2, there exists $\overline{x} \leq x^*$ such that for $x < \overline{x}$ we have

$$f(x - c(x), a^*) \geq x.$$

We can write

$$\log x_{t+1} = \log f(x_t - c_t, a_{t+1}) - \log f(x_t - c_t, a^*) + \log f(x_t - c_t, a^*) - \log x_t + \log x_t.$$

Since $u_{t+1} \leq \log[f(x,a)] - \log[f(x,a^*)]$ when $x < x^*$, we obtain for all $x < \overline{x}$ that

$$\log x_{t+1} \geq u_{t+1} + \log x_t.$$

For $x \geq \overline{x}$ we have

$$\log x_{t+1} \geq \log f(\overline{x} - c(\overline{x}), a_{t+1}).$$

Let $\nu_{t+1} = \log f(\overline{x} - c(\overline{x}), a_{t+1}) - u_{t+1}$, then

$$y_{t+1} = u_{t+1} + \min[y_t, \nu_{t+1}]$$

is a process that is dominated by $\log x_{t+1}$. Applying Proposition 2 (i), noting that we need to ensure y_{t+1} does not escape to minus infinity, it follows that y_{t+1} satisfies asymptotic tightness, which implies that x_{t+1} does not escape to minus infinity either.

To show that x_t satisfies asymptotic tightness from above, note that $\log f(x,a) - \log x$ is monotone decreasing in x for all a, and $\lim_{x\to\infty}[\log f(x,a) - \log x] = -\infty$. Since $\log f(x,a)$ is integrable, it follows from the dominated convergence theorem that there exists $\overline{x}$ such that $\mathrm{E}\log f(\overline{x},a) < \log \overline{x}$. Let $u_{t+1} = \log f(\overline{x}, a_{t+1}) - \log \overline{x}$.

Now consider

$$\log x_{t+1} \leq \log f(x_t, a_{t+1}) - \log x_t + \log x_t.$$

For all $x > \overline{x}$ we have

$$\log x_{t+1} \leq u_{t+1} + \log x_t$$

while for all $x \leq \overline{x}$ we have

$$\log x_{t+1} \leq \log f(\overline{x}, a_{t+1}).$$

Let $\nu_{t+1} = \log f(\overline{x}, a_{t+1}) - u_{t+1}$, then

$$y_{t+1} = u_{t+1} + \max[\log y_t, \nu_{t+1}]$$

clearly dominates the dynamics of $\log x$, and satisfies asymptotic tightness by Proposition 2 (i). As a result, so does x_{t+1}.

The fact that x is increasing follows as in Hopenhayn and Prescott (1992). I now turn to mixing. Let $\underline{a}$ denote the infimum of the support of a, and begin by assuming that $\underline{a} = 0$. The above argument establishing tightness showed that for any initial condition x_0, the process x_{t+1} will eventually assume a value bounded away from zero with positive probability. Since $\underline{a} = 0$, we also have that for any initial condition x_0, the process x can assume a value arbitrarily close to zero in one step, and these two observations imply mixing.

If $\underline{a} > 0$ then mixing can be shown by slightly modifying Hopenhayn and Prescott's argument. Let k^* be the unique solution to the equation $1 = \beta \mathrm{E} f'(k^*, a)$, and let x^* be the unique value such that $x^* - c(x^*) = k^*$. Let x^0 be large enough that $f\left(x^0 - c(x^0), \underline{a}\right) < x^0$, and define the sequence $x^{n+1} = f(x^n - c(x^n), \underline{a})$. This is a decreasing and bounded sequence which converges to some value $\overline{x}$. By continuity, $f(\overline{x} - c(\overline{x}), \underline{a}) = \overline{x}$. The first order condition together with an envelope theorem implies

$$u'(c(\overline{x})) = \beta \mathrm{E}\left[f'(\overline{x} - c(\overline{x}), a) \cdot u'(c(f(\overline{x} - c(\overline{x}), a)))\right].$$

We have $f(\overline{x} - c(\overline{x}), a) \geq f(\overline{x} - c(\overline{x}), \underline{a}) = \overline{x}$ and hence

$$u'(c(\overline{x})) < \beta \mathrm{E}\left[f'(\overline{x} - c(\overline{x}), a) \cdot u'(c(\overline{x}))\right]$$

where the inequality is strict if a is non-degenerate. This implies $1/\beta <\mathrm{E}[f'(\overline{x} - c(\overline{x}), a)]$ or $\overline{x} < x^*$. Since $\underline{a}$ is the infimum of the support of a, the probability that x_t will be in any neighborhood of $\overline{x}$ eventually is positive.

Now consider an arbitrary low value of x^0, and define the sequence $x^{n+1} = f(x^n - c(x^n), \overline{a})$. An analogous argument shows that for x^0 low enough, we eventually get to a point $\underline{x}$ such that $\underline{x} > x^*$. This shows that x is mixing with respect to x^*.

It thus follows that x has a stable distribution. To show that this distribution is also invariant, I note that with small modifications, the above arguments can also be used to show that the upper and lower bound processes also satisfy weak mixing. Because they are Feller, they possess invariant distributions, and hence so does x.

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