Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 139 (2008) 295–309
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Core and periphery in networks

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Received 31 January 2007; final version received 31 July 2007
Available online 22 September 2007

Abstract

We study a model of network formation where the benefits from connections exhibit decreasing returns and decay with network distance. We show that the unique equilibrium network is a periphery-sponsored star, where one player, the center, maintains no links and earns a high payoff, while all other players maintain a single link to the center and earn lower payoffs. Both the star architecture and payoff inequality are preserved in an extension of the model where agents can make transfers and bargain over the formation of links, under the condition that the surplus of connections increases in the size of agents' neighborhoods. Our model thus generates two common features of social and economic networks: (1) a core-periphery structure; (2) positive correlation between network centrality and payoffs.

JEL classification: A14; C72

Keywords: Social networks; Social capital; Network formation; Centrality

1. Introduction

Social networks provide individuals access to various resources, including information, knowledge, emotional support, financial support, and insurance. [1] The extent to which these resources are available depends in part on the shape of the network and the position an agent occupies in it. Two empirical regularities about network structure, access to resources and payoffs motivate this paper. First, many economic and social networks exhibit a *core–periphery structure*: A small

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[1] Connections are an important source of information about jobs (Granovetter [11]), consumption products (Reigen and Kernan [20]), health alternatives (Feldman and Spencer [7]), and investment opportunities (Hong et al. [13]).

0022-0531/$ - see front matter
doi:10.1016/j.jet.2007.07.007

number of central agents or "hubs" gather a disproportionate amount of connections, while most other agents maintain few relationships. [2] Second, there is a positive correlation between network centrality and various measures of payoffs. [3] This paper presents a model of network formation that exhibits both of these features in a stylized way.

Our model has two key assumptions: (1) the benefits from accessing others have strongly decreasing returns to scale; and (2) benefits decay with network distance. We illustrate the setup and assumptions with an example where the social network is a basis for information exchange. Agents have imperfect information about the state of the world, but they can use the social network to obtain noisy signals about the information of others. The benefits from connections exhibit decreasing returns: with a large enough number of connections, access to additional people has a small effect in reducing uncertainty. Moreover, frictions in communication imply that benefits decrease with the distance between players.

The main result of the paper is that under our two key assumptions, the network formation model has a unique Nash equilibrium network architecture: the *periphery-sponsored star*. In this equilibrium there is a single player, called the center, who maintains no links, and all other players maintain one link to the center. The basic logic for this result is straightforward. Frictions in communication induce agents to form links that keep them close to one another. As a result, in any equilibrium there must exist agents who have a large number of close connections. In such a network, due to strong decreasing returns, all agents find it optimal to maintain at most one link: connecting to a well-connected individual provides access to a high payoff, which makes forming a second link suboptimal. In the periphery sponsored star, the center earns the highest payoff, both because he has the most direct access to all agents in the economy, and because he does not maintain any links. As a result, the equilibrium in our model exhibits both of the motivating stylized facts described above.

One limitation of our basic model is that the cost of any connection is fully borne by the agent maintaining the link, and thus connections do not require mutual consent. To address this limitation, we allow agents to make transfers and bargain over the formation of links as in Bloch and Jackson [4]. In this context, the incentives of two players to form a link are determined by the surplus generated by that link. We identify a key additional condition under which the star architecture continues to be the unique equilibrium network: Loosely, the surplus from connecting to an agent must be monotonically increasing in the number of his neighbors. [4] When this condition holds, any competition between agents who aspire to be centers tips in favor of a single agent because the incentives to connect to that candidate increase as his neighborhood grows.

This paper builds on the network formation literature in economics that followed Jackson and Wolinsky [15]. The paper most closely related to our work is Bala and Goyal [2]. They study a non-cooperative network formation model and find that in the absence of communication frictions there are a large number of pure strategy Nash equilibria, but there exists a unique strict Nash equilibrium architecture, the center-sponsored star. They also show that for small frictions,

[2] Examples include networks of scientific collaboration (Newman [18]; Goyal et al. [10]), friendship (Adaic and Adam [1]), advice in the workplace (Cross et al. [6]), and interlocking directorates of corporations (Mizruchi [17]).

[3] Centrality is positively associated with higher academic performance (Balwin et al. [3]), finding better paid jobs (Lin [16]), promotion rates and evaluation (Burt [5]); and firm performance (Powell et al. [19]).

[4] The analogue of this condition always holds in the basic model without transfers.

the periphery-sponsored star is the unique strict Nash equilibrium.[5] Feri [8] studies an evolutionary version of the Bala and Goyal model with linear payoffs and exponential decay and shows that for some parameters, stochastically stable networks are periphery-sponsored stars. Our model qualifies and extends these results in several directions. First, we do not need to rely on strictness or stochastic stability for equilibrium selection because Nash equilibrium already gives a unique prediction. Second, the center-sponsored star in which the center obtains the lowest payoff in the network is never an equilibrium under our assumptions. Third, we obtain the periphery-sponsored star for a large class of benefit functions with arbitrary decay, not just small communication frictions or exponential discounting. Finally, we show how to extend our results to an environment where agents can make transfers and bargain over links. Our relatively sharp equilibrium characterization compared to the earlier literature follows from the interaction of the assumptions on decreasing returns in benefits and decay in communication.

2. A model of network formation

Consider a simultaneous-move game in which N players decide on maintaining links to one another. We also denote the set of agents by N. A strategy for player $i \in N$ is a vector $s^i \in \{0, 1\}^{N-1}$, where $s^i_j = 1$ if player i decides to link j and 0 otherwise. The set of links formed by all agents define an undirected graph or network denoted by $g = (s^1, \dots, s^N)$ where agents are the nodes and links are the edges.

Once in place, each link provides access to others in both directions. Direct and indirect access to other agents in the network generates benefits, which can depend on the network distance between players. To formalize this idea, define the distance $d\,(i, j)$ between players i and j to be the number of edges along the shortest path between i and j in g.[6] For each player i, denote the number of links i maintains by $l^i = \sum_{j \in N} s^i_j$, and denote the number of players who are exactly at distance k from i in the network g by n^i_k. The payoff of player i in the network formation game is defined as

$$\pi^i(s^i, s^{-i}) = f(a_1 \cdot n^i_1 + a_2 \cdot n^i_2 + \cdots + a_d \cdot n^i_d) - c \cdot l^i, \tag{1}$$

where $f(.)$ is an increasing benefit function and $c > 0$ is the cost of forming and maintaining a link. Here d is the *communication threshold*: agents who are more than d far away yield no benefit. The positive weights $a_1, a_2, \dots, a_d$ measure the relative importance of neighbors at different distances. We assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d$, so that more distant neighbors yield weakly less benefits. We also normalize $a_2 = 1$. This assumption is inconsequential, but makes it easier to state our results. Useful special cases are (1) $a_1 = a_2 = \cdots = a_d = 1$ in which case direct and indirect neighbors are equally important; and (2) $a_{s+1} = \beta \cdot a_s$ with $0 < \beta < 1$, that is, geometric discounting of benefits by distance. Below, we sometimes refer to the sum $\overline{n}^i = a_1 \cdot n^i_1 + a_2 \cdot n^i_2 + \cdots + a_d \cdot n^i_d$ as the number of *effective neighbors* that i has access to.

2.1. Key assumptions

Throughout the paper, we make the following two assumptions about the payoff structure.

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[5] Galeotti et al. [9] extend this model to allow for heterogeneity in the cost of links and obtain periphery-sponsored interlinked stars as the unique strict-Nash equilibrium with small frictions in communication.

[6] If no such path exists, the distance is set to infinity.

Assumption 1 (*Strong decreasing returns*). The benefit function $f(.)$ is strictly increasing, concave, and there exists $M \geqslant 0$ such that for all $m > M$

$$f(2m) - f(m) < c. \tag{2}$$

Functions satisfying this property are said to exhibit (M, c)-strong decreasing returns. When $f(.)$ is bounded, for any c there exists an M such that $f(.)$ exhibits (M, c)-strong decreasing returns; thus, standard utility functions, such as exponential utility, or power utility with risk aversion coefficient exceeding one, satisfy Assumption 1 with some M. Intuitively, $f(.)$ exhibits strong decreasing returns if it grows slower than the logarithm function.[7] Note that if the weights $a_1, \ldots, a_d$ are proportionally changed, then both $f(.)$ and the threshold M in the definition of strong decreasing returns need to be modified accordingly. This is the reason for normalizing $a_2 = 1$.

Assumption 2 (*Limits to communication*). The communication threshold d is finite.

2.2. An example: network formation and the value of information

Assumption 2 seems plausible in many contexts. To motivate Assumption 1 and our general model setup, consider an economy where agents exchange information about a single unobserved state variable $\theta \sim N(\mu, \tau^2)$. Agents wish to learn about θ because they have to choose an action $x^i \in \mathbb{R}$ that maximizes expected utility $\mathrm{E}\left[-(x^i - \theta)^2\right]$. Once θ is drawn, each player i receives a signal $z_i \sim N(\theta, \sigma^2)$, where signals are conditionally independent across agents.

Agents exchange information with neighbors in the network. The transmission of information is subject to frictions: if player i is at a distance of $d(i, j) \leqslant d$ from player j, then i receives a signal from j which equals $\tilde{z}^i_j = z_j + \sum_{k=1}^{d(i,j)} \varepsilon^i_{kj}$. Here z_j is player j's original signal, and the sum of the ε^i_k terms is the cumulative noise of transmission. The ε^i_{kj} are i.i.d. and normally distributed with mean zero and variance σ^2_ε. This reduced form can be obtained if there are d rounds of communication, where in each round players communicate with direct neighbors and transmit noisy versions of all the signals they observe.

Given the normality assumption and quadratic preferences, the expected utility of agent i in a given network can be computed explicitly as

$$\frac{-1}{\pi + \rho\left(1 + \sum_{\delta=1}^{d} a_\delta n^i_\delta\right)} = \frac{-1}{\pi + \rho + \rho \bar{n}^i}, \tag{3}$$

where $\pi = 1/\tau^2$ and $\rho = 1/\sigma^2$ are the precisions of the state and the signal, $a_\delta = \sigma^2 / \left(\sigma^2 + \delta\sigma^2_\varepsilon\right)$, and $\bar{n}^i$ is the number of i's effective neighbors. If $\sigma^2_\varepsilon = 0$ then $a_\delta = 1$ for all $\delta \leqslant d$, while $\sigma^2_\varepsilon > 0$ implies that a_δ is strictly decreasing in δ: distant neighbors contribute less to the precision of the signal due to accumulated noise. This payoff function is bounded, and hence satisfies (M, c) -strong decreasing returns for some M.[8] This economy with information exchange can be viewed as one possible microfoundation for our model setup and for the strong decreasing returns assumption.

[7] Note that $f(n) = A \cdot \log(n)$ has $(0, c)$-strong decreasing returns as long as $A < c/\log 2$.

[8] It can be shown that (M, c)-strong decreasing returns is satisfied with $M = (1/2\rho)\left[1/c - (3/2)\,\pi + \left\{((1/c) - (3/2)\,\pi)^2 - 2\pi^2\right\}^{1/2}\right]$ if $c \leqslant \left[(\sqrt{2} + 3/2)\pi\right]^{-1}$ and with $M = 0$ otherwise.

3. Equilibrium networks

Our goal is to determine the pure strategy Nash equilibria of the network formation game described above. We begin by introducing some graph theoretic concepts. Two agents are *connected* to each other if there exists a path or sequence of links in the network between them. A *component* of the network is a maximal set of connected agents. The network is connected if all agents are connected to each other. The network in which no player maintains a link is called the *empty network*. An equilibrium is *non-empty* if at least one player maintains a link. An agent i is called the *center* of a set of agents S if $i \in S$ and all $j \in S$, $j \neq i$ maintain a link to i. The network architecture is a periphery-sponsored star if there exits a player i who is the center of all N agents, and there are no links other than those agents in $N \backslash \{i\}$ maintain to i. The network is an *extended star* some agent i maintains no links, all other players maintain a single link and are directly or indirectly connected to i, and $d(j, k) \leqslant d$ for all $j, k \in N$.

3.1. Main result

Recall that $f(.)$ is assumed to exhibit (M, c)-strong decreasing returns, and let $N_0 = (2M)^{2d+2}$.

Theorem 1. *If* $a_2 > a_3$ *and* $N > \max(N_0, 4)$ *then the unique non-empty equilibrium architecture is a periphery-sponsored star. If* $a_2 = a_3$ *then for* $N > \max(N_0, 2d + 1)$ *any non-empty Nash equilibrium is an extended star.*

The proof proceeds using a series of lemmas. First we establish that in equilibrium the network is tight: no two players are very far from each other. This result is a consequence of Assumption 2 about limits to long distance communication. Tightness implies that there exist players who have many direct neighbors. Next we establish that in a large network, each player maintains at most one link. This "one-link property" follows because agents can access substantial benefits by forming a single link to some player with many direct neighbors. Since the benefit function exhibits strong decreasing returns, maintaining a second link is not optimal. Building on the one-link property, we use graph-theoretic arguments to show that in equilibrium the network is either a tree, or contains a unique directed cycle. We complete the proof using a revealed preference argument that is based on comparing the payoffs of agents to whom no one links.

Define the k-neighborhood of an agent i as $N_k(i) = \{j \in N | d(i, j) \leqslant k\}$, i.e., the set of agents who are within distance k from i, and let $|N_k(i)|$ be the number of agents in i's k-neighborhood.

Lemma 1. *In any non-empty equilibrium no agent is isolated, and there exists a player who is at most* $2d + 1$ *far away from any other player.*

Proof. Non-emptiness implies that some $l \in N$ maintains some links: $\sum_j s_j^l > 0$. Now suppose that k is isolated, and consider the deviation where k forms links to all agents j for whom $s_j^l > 0$, i.e., whom l maintains links with. Concavity of $f(.)$ implies that the benefit to k of these links links will be at least as high as it is to l. Since l is behaving optimally, these benefits must at least cover the cost of links. Moreover, k also gains indirect access to l; as a result, maintaining these links would be strictly beneficial to him, contradicting the assumption that k is isolated in equilibrium.

Let i be one agent for whom the sum $a_2 \cdot n_1^i + a_3 \cdot n_2^i + \cdots + a_d \cdot n_{d-1}^i$ takes its maximal value in the network. Suppose there exists an agent who is more than $2d+1$ far from i. Since this agent is not isolated, she either maintains a link, or is linked to by someone. In both cases, there is a $j \in N$ who maintains a link and for whom $d(i, j) \geqslant 2d+1$. Now j will find it beneficial to drop one of her links and instead form a new link to i. To see why, note that $d(i, j) \geqslant 2d+1$ implies $N_{d-1}(i) \cap N_d(j) = \emptyset$. Thus, prior to the deviation, j enjoys no benefits from agents in $N_{d-1}(i)$. Connecting to i would then generate $a_1 + a_2 \cdot n_1^i + a_3 \cdot n_2^i + \cdots + a_d \cdot n_{d-1}^i$ in effective connections to j. On the other hand, giving up the link to k would reduce j's effective connections by strictly less than $a_1 + a_2 \cdot n_1^k + a_3 \cdot n_2^k + \cdots + a_d \cdot n_{d-1}^k$, because this latter sum contains agent j himself. Since $a_2 \cdot n_1^i + a_3 \cdot n_2^i + \cdots + a_d \cdot n_{d-1}^i \geqslant a_2 \cdot n_1^k + a_3 \cdot n_2^k + \cdots + a_d \cdot n_{d-1}^k$ by construction, the deviation is strictly profitable for j. □

Lemma 2. *For $N > N_0$, in any non-empty equilibrium all players maintain at most one link.*

Proof. If all players have at most u direct neighbors, then $N \leqslant u^{2d+2}$ because no agent is more than $2d+1$ away from the maximal player of Lemma 1. Reorganizing this condition shows that there must be some $i \in N$ for whom $|N_1(i)| \geqslant N^{1/(2d+2)}$. Then any $j \in N$ who maintains a link must have access to at least $a_2 \cdot N^{1/(2d+2)} = N^{1/(2d+2)}$ effective neighbors, because j can choose to drop all links and connect to i, which provides her with at least $a_2 \cdot N^{1/(2d+2)}$ effective neighbors.

Suppose there exists a player i who maintains more than one link. Let L be the set of players i is linking to, m the total number of effective neighbors she has access to, and for each $j \in L$, let m_j be the effective number of people that i has access to only through j. Clearly, $\sum_{j \in L} m_j \leqslant m$. Because L has at least two elements, there is a j such that $m_j \leqslant m/2$. But then dropping the link to j is surely profitable if $\max_{N^{1/(2d+2)} \leqslant m} (f(m) - f(m/2)) < c$, which holds by strong decreasing returns. □

The "one-link property" is useful because it restricts the architecture of the network. More precisely, a non-empty network with this property is either a directed tree, or contains a unique directed cycle such that agents in the cycle are the endpoints of disjoint directed subtrees. [9] To see the logic, assume that there is some player i who maintains no links. Then the agents in $N_1(i)$ must all have a single link to i and maintain no other links. Hence, each agent in $N_2(i) \backslash N_1(i)$ must maintain a single link to somebody in $N_1(i)$. Repeating this argument for more and more distant neighbors of i leads to the directed tree network. A directed cycle arises if there is no agent i who maintains no links. The cycle can be found simply by following any directed path in the network: since all agents maintain a link, the path can only stop when a cycle is found. Given the cycle, the disjoint directed trees can be identified by looking at more and more distant neighborhoods of each agent in the cycle, as in the argument above.

We proceed by examining the incentives of *terminal nodes*, i.e., agents with no incoming links.

Lemma 3. *Suppose that $N > N_0$, and let $i, j \in N$ be terminal nodes in a non-empty equilibrium. Then $a_2 = \cdots = a_{d(i,j)}$ and $d(i, j) \leqslant d$. In particular, if $a_2 > a_3$ then $d(i, j) = 2$ and all terminal nodes maintain a single link to the same player.*

[9] An (i, j) link is said to be directed from i to j if i maintains it, i.e., if $s_j^i = 1$. A network is a directed tree if it is a tree ignoring the link directions, and all links are directed toward a particular agent. A directed cycle is a closed directed walk with no repeat vertices except for the starting and ending vertices.

Proof. Since there are no isolated players, both i and j maintain a single link. Suppose that $\overline{n}^i \leqslant \overline{n}^j$ so that i has fewer effective neighbors and lower payoff, and consider the deviation where i drops his single link and maintains a link to j's direct neighbor instead. By doing so, i will access $\overline{n}^j + a_2 - a_{d(i,j)}$ effective neighbors. This is because (1) i will access $n_2^j + 1$ agents at distance 2, where the $+1$ represents player j; (2) we need to subtract $a_{d(i,j)}$ because $\overline{n}^j$ included the benefit of j from indirect access to i. This deviation is profitable as long as $a_2 - a_{d(i,j)} > 0$. Hence, $a_2 = \cdots = a_\delta$ and, since $a_2 > 0$ and $a_{d+1} = 0$, $d(i, j) \leqslant d$. If $a_2 > a_3$ this requires $d(i, j) = 2$. □

The lemma implies that in the case where $a_2 > a_3$, the network consists of a star, a single directed path and a directed cycle. In the Appendix we show that the path and the cycle are of zero length:

Lemma 4. *If $N > \max(N_0, 4)$ and $a_2 > a_3$, then the unique non-empty Nash equilibrium architecture is a periphery-sponsored star.*

This concludes the proof of Theorem 1 when $a_2 > a_3$. If $a_2 = a_3$, then we have

Lemma 5. *If $a_2 = a_3$ and $N > \max(N_0, 2d + 1)$, then a non-empty equilibrium is an extended star. Moreover, if the maximum distance in the equilibrium is $\overline{d}$, then it must to be that $a_2 = \cdots = a_{\overline{d}} = 1$.*

The proof, given in the Appendix, uses Lemma 3 to show that any two players in the network are at most d far from each other. We then only need to rule out the possibility of a directed cycle. We show that in a directed cycle, there always exists a player who benefits more from her incoming links than from the link she maintains; but then he should drop his link under strong decreasing returns, a contradiction. This concludes the proof of Theorem 1.

Discussion of main result: Three remarks about the theorem are in order. First, the empty network is an equilibrium if and only if $f(a_1) - f(0) \leqslant c$, because otherwise players will wish to form at least one link. Second, the periphery-sponsored star is an equilibrium when $f(a_1 + (N-2)a_2) - f(0) \geqslant c$. This condition ensures that the periphery-sponsored star is an equilibrium even if Assumptions 1 and 2 fail, as long as f is increasing, (weakly) concave, and satisfies $f(2a_1 + (N-3)a_2) - f(0) \leqslant 2c$.[10] Thus the key content of our assumptions is that they guarantee uniqueness of the equilibrium network. Finally, uniqueness can be obtained even without limits to communication if we require that f satisfies $(1, c)$-strong decreasing returns. Under this condition, any player maintaining two or more links will drop his least useful link, because $f(b) - f(b/2) < c$ holds for any $b \geqslant 2$. This establishes the "one-link property" directly, and then Lemmas 3 and 4, which do not rely on Assumption 2, can be used to show uniqueness of the periphery-sponsored star as long as $a_3 < a_2$.

3.2. Payoffs and welfare

In the periphery-sponsored star, the center has a payoff advantage relative to a player in the periphery of $[f((N-1)a_1) - f(a_1 + (N-2)a_2)] + c$. The first term measures the additional payoff the center derives from having *direct* access to the entire population. This term is large

[10] The last condition follows from $f(a_1 + (N-2)a_2) - f(0) \geqslant c$ when Assumption 1 holds.

when a_1/a_2 is large, i.e., when the communication technology is poor. The second term derives from the fact that the center maintains no links. This term is small when link costs are small.

Is the inequality in the equilibrium network associated with inefficiency? Since players are ex-ante identical, it is natural to use a symmetric utilitarian welfare function: $W(g) = \sum_{i=1}^{N} \pi^i(g)$. Because the costs of links are separable, $W(g)$ is not affected by how the costs of links are distributed. Jackson and Wolinsky [15] show that when $f(.)$ exhibits constant returns to scale, for intermediate costs of connections the star is the only efficient architecture. With decreasing returns the results can be quite different:

Example 1. Suppose that N is even and let $f(n) = -1/(1+n)$, $c = 1$. Then the unique efficient network architecture has $N/2$ components, each with two players. The average payoff in the efficient network exceeds average payoff in the star by a term bounded away from zero for all N.

For an intuition, observe that in the network with $N/2$ components, the per capita link cost is only $\frac{1}{2}$, while in a connected network it would be $(N-1)/N$. When $f(.)$ is sufficiently concave, as in the example, the benefits provided by the network do not grow quickly enough to compensate for the higher per capita cost of links, which makes networks with few links more efficient. This reasoning suggests that the efficiency of the star requires a lower bound on the curvature of $f(.)$.

A second source of inefficiency for the star is the inequality in the number of effective neighbors across agents. Computing $W(g)$ requires averaging the concave function $f(.)$, and this yields a higher value when the arguments are less unequal. Based on this logic, it is easy to construct examples where welfare in the star is smaller than welfare in a network with a more equal distribution of effective neighbors, such as two interlinked stars.

The next result shows that concavity and inequality are the only two sources of inefficiency of the star.

Proposition 1. *Assume that $f'(n) > 2ca_1^2/n^2$ for $n > a_1$ and that $N > f^{-1}(c + f(0))$.*

(i) *If $a_1 = a_2$ and $a_3 = 0$, the unique efficient network architecture is the star.*
(ii) *For $N > \max\{N_0, 3a_1 - 2\}$ the per capita payoff advantage of the efficient network relative to the star is at most $2c(1 - 1/a_1)/N$, and the number of components in an efficient network is bounded by a constant independently of N.*

Part (i) says that when $f(.)$ is not too concave and payoff inequality is limited by $a_1 = a_2$, the star is efficient. For an intuition, note that in a star of size m, the per capita link cost is $cm/(m-1)$, which has a slope in m of order c/m^2. If the benefits grow at some slower rate, then it is more efficient to split agents into smaller stars than to keep them together in a single component. Part (ii) says that the equilibrium is approximately efficient for N large. This is a consequence of strong decreasing returns, which make the benefit function relatively flat in the limit.

4. Transfers

We now extend our basic network formation game to include transfers. Incorporating transfers allows us to study of competition between different agents who aspire to be centers. Our goal here is to explore conditions under which the star continues to be the unique equilibrium network.

Let the strategy of player i be a vector $t^i \in \mathbb{R}^{N-1}$, where t^i_j is the amount that i is offering to pay for a link between i and j. An (i, j) link will be formed if and only if $t^i_j + t^j_i \geqslant c$. When

$t^i_j + t^j_i > c$, the additional resources beyond c are wasted. The profile t defines a network $g(t)$, and the payoff of i is

$$\pi^i(t^i, t^{-i}) = f(a_1 n^i_1 + a_2 n^i_2 + \cdots + a_d n^i_d) - \sum_{j:\,\exists \text{link}(i,j)} t^i_j. \tag{4}$$

We refer to this network formation game as the transfers game.[11] A key feature of the transfers game is that players can reject links: e.g., i can set t^i_j to be a large negative number, preventing the link from being formed even if j is willing to pay the entire cost c or more.

In models where creating links requires mutual consent, connections that benefit both parties need not be formed in a Nash equilibrium. Jackson and Wolinsky [15] propose the refinement of pairwise stability to deal with this problem.[12] With strong decreasing returns, pairwise stability is a relatively weak requirement, because additional links are often costly, and hence a new (i, j) link may be of mutual interest only when i or j can sever some existing links. This leads to the following definition:

Definition 1. A profile t is a pairwise Nash equilibrium with a unilateral status quo if for any action $\widetilde{t^i}$ of i and action $\widetilde{t^j}$ of j, where $\widetilde{t^j_k} = t^j_k$ for all $k \neq i$, $\pi^i(\widetilde{t^i}, \widetilde{t^j}, t^{-i-j}) > \pi^i(t^i, t^{-i})$ implies $\pi^j(\widetilde{t^j}, \widetilde{t^i}, t^{-i-j}) < \pi^j(t^j, \widetilde{t^i}, t^{-i-j})$.

Beyond Nash equilibrium, this definition requires t to be robust to all deviations where i makes arbitrary changes to t^i, proposes a link to j by modifying the transfer t^j_i, and j is better off accepting the link given the changes i made in her other links. Here j, when deciding on accepting the (i, j) link, assumes that i has already implemented the other changes in her strategy, and hence computes the potential gain of accepting relative to the profile $(t^j, \widetilde{t^i}, t^{-i-j})$. Thus the outside option or "status quo" payoff of j is computed in a network where i may have severed some of his links. Such outside options are sensible if it is costly to re-establish a severed connection, or when agents are naive and do not question the credibility of implausible threats.

In the analysis below, we only focus on the case where $f(.)$ satisfies $(1, c/2)$-strong decreasing returns. This assumption implies a variant of the "one-link property" irrespective of population size. Our results highlight the importance of the following additional restriction on payoffs.

Definition 2. If the function $f(a_1 + n) + f(a_1(n+1)) - f(a_1 n)$ is increasing in n for $n \geqslant 1$, then we say that $f(.)$ satisfies the monotone surplus condition (MS) with respect to a_1.

To understand this condition, recall that in our basic network formation game, a player choosing between connecting to one of two separate stars always finds it optimal to maintain a link to the center of the larger one. In the transfers game this need not hold, because here the choice to form a link is determined by the surplus created by the link for the two parties. A link with the center of the smaller star may create higher surplus if that center benefits substantially from an additional neighbor. Formally, the surplus from a link between the isolated player and the center of a star with n neighbors is $f(a_1 + a_2 n) - f(0) + f(a_1(n+1)) - f(a_1 n) - c$, where the first two terms are the gain of the previously isolated player, the next two terms are the gain of the center, and the last

[11] This protocol is equivalent to the direct transfer network formation game of Bloch and Jackson [4].

[12] See Jackson [14] for a discussion.

term is the cost of the new link. The surplus from connecting to a center with more neighbors will be higher when this expression is monotone increasing in n, which is exactly condition (MS). [13]

Proposition 2. *Suppose that $a_{\delta+1}/a_{\delta}$ is weakly decreasing for $1 \leqslant \delta \leqslant d-1$ and $N > 4d-1$. If $f(.)$ satisfies $(1, c/2)$ strong decreasing returns and* (MS) *then any non-empty pairwise equilibrium with unilateral status quo is a star where each player in the periphery transfers $c/2$ or more to the center. Moreover, if $N \geqslant 3 + 1/(a_1 - 1)$ than the center earns the highest payoff.*

This result shows that when (MS) holds, the star continues to be the unique equilibrium network. The proof proceeds by first assigning directions to the links in the network: An (i, j) link is said to be directed from i to j if $t_j^i \geqslant c/2$. Using $(1, c/2)$-strong decreasing returns, it can now be shown that all players have at most one link originating from them. The proof then analyses the incentives of terminal nodes. If two terminal nodes i and j have links with two different "centers" k_i and k_j, and k_i offers more effective neighbors than k_j, then (MS) guarantees that k_i would always be able to attract j by requiring the latter to pay a lower transfer. Thus all terminal nodes are connected to the same agent, and the rest of the proof then follows easily.

5. Conclusion

This paper develops a model of network formation where communication has frictions and the benefits of connections exhibit strong decreasing returns. These two assumptions lead to a unique equilibrium architecture that exhibits a core–periphery structure as well as a positive correlation between centrality and payoffs. These results continue to hold if agents are allowed to make transfers over links, as long as connecting to more centrally located agents generates higher surplus. In our working paper, Hojman and Szeidl [12], we show how to extend the model to study equilibria with heterogenous social groups and the selection of the center when agents have different abilities.

Acknowledgments

We are grateful to Attila Ambrus, Ed. Glaeser, Sanjeev Goyal, Kenneth Judd, Larry Katz, Markus Mobius, Nolan Miller, Jesse Shapiro, Jeremy Tobacman, Richard Zeckhauser, an associate editor and two referees, seminar participants, and especially Drew Fudenberg and Jerry Green for helpful comments.

Appendix. Proofs

Lemma 6. *Suppose the equilibrium network has three players i, j and k such that* (i) *agent i maintains a link with j and j maintains a link with k;* (ii) $N_1(j) = \{i, k\}$; *and* (iii) *for some player l, $d(k, l) + 1 < d(i, l)$. Then $a_2 = a_3 = \cdots = a_{d(i,l)}$ and $d(i, l) \leqslant d$.*

Proof. Let $S = \{u | d(i, u) > d(k, u) + 1\}$. For all agents $u \in S$, one shortest path between i and u runs through k, because combining the path i, j, k with the shortest path between k and u gives a path between i and u of length $d(k, u) + 2$, and by definition of S we have $d(i, u) \geqslant d(k, u) + 2$.

[13] An example where (MS) holds is $f(n) = -A/(k+n)$, as long as $a_1 < k^2$. This $f(n)$ also satisfies $(1, c/2)$ strong decreasing returns when $2Ac < (2 + 1/k)(1 + 1/k)$.

By (iii) we have $l \in S$; it is also easy to check that if $l_1, \ldots, l_{d(k,l)} = l$ is the shortest path connecting k to l, then $l_t \in S$ too. Now consider the deviation where i drops her link with j and links k instead. The size of $N_1(i)$ under the deviation remains the same: k replaces j. By (ii), the set $N_2(i) \setminus N_1(i)$ loses k, but gains j, so its size is also weakly increasing. All agents in $N \setminus S$ continue to be at the same distance from i as before, since the shortest path between i and these agents is unaffected by the deviation. Finally, all remaining agents $u \in S$ are one link closer to i, because the deviation reduces by one the length of the shortest path between u and i, which runs through k. Each agent $u \in S$ at distance $d(u, i)$ before the deviation will improve the payoff of i by $a_{d(u,i)-1} - a_{d(u,i)}$. Since there are agents in S at all distances $3, \ldots, d(i, l)$ from i, and the deviation cannot be profitable, it must be that $a_2 = a_3 = \cdots = a_{d(i,l)}$ and, in particular, $d(i, l) \leqslant d$. □

Proof of Lemma 4. From Lemmas 2 and 3 the network consists of a star, a directed path starting from the center of this star—player i—and a directed cycle at the end of this path. Since $a_2 > a_3$, Lemma 6 implies that there can be no intermediaries between the center and some player in the cycle, i.e., that the path must have length at most one: otherwise, we could apply the Lemma for a contradiction on a chain of agents i, j and k, where j is in the path and k may be in the cycle. Similarly, Lemma 6 also implies that the cycle can have at most three players.

We now show that if $N > 4$ and either the path or the cycle have positive length, then there is always a player with a profitable deviation. If there is no cycle and the path has length one, then i maintains a link that gives him access to a single player. Thus i has $N - 2$ incoming links, and by strong decreasing returns it is optimal for him to drop his link. The same logic holds if the cycle has three agents and i is one of them: one of the other two agents in the cycle has a link to i; thus i has $N - 2$ incoming links and should maintain no links. If the cycle has two players, then one of them is maintaining a redundant link which should be dropped in equilibrium. Finally, if the cycle has three players and the path has length one, i.e., i is outside the cycle and links to some j in the cycle, then the player in the cycle who is also linking player j is better off dropping her link and linking i instead. □

Proof of Lemma 5. We start by showing that the maximal distance $\overline{d} \leqslant d$ and that $a_2 = a_3 = \cdots = a_{\overline{d}}$. By the discussion after Lemma 2, the equilibrium network contains at most one directed cycle. Since $N > 2d + 1$, by Lemma 6, there must be players in the network that are not in the cycle and hence there must be some terminal nodes. Thus the maximal distance in the graph, $\overline{d}$, is achieved by a terminal node t and some other player i. If i is a terminal node, $a_2 = a_3 \cdots = a_{\overline{d}}$ follows from Lemma 3. Otherwise, i must belong to the directed cycle. In this case, i maintains a link with a player j in the cycle whose only incoming link is i; otherwise, j would be the root of a directed tree and the distance from t to any player in such tree—including the terminal nodes—would be at least $\overline{d}$. Since player j must be forming a link with some other player k in the cycle, we can apply Lemma 6 with $l = t$ to conclude $a_2 = a_3 = \cdots = a_{\overline{d}}$.

We next show that the equilibrium network does not contain a cycle. Otherwise, we show that there is a player i in the cycle who gets more effective neighbors from her incoming link along the cycle than from the link she maintains. If this holds, by $a_2 = a_3 = \cdots = a_{\overline{d}}$, we know that i has at least $N - 1$ effective neighbors, and then strong decreasing returns implies that she should drop her link. To see why such a player i exists, for each player j in the cycle let $w^{jk} = a_\delta$ if the distance $\delta = d(j, k)$ between j and k is attained by a path that contains the link maintained by j; otherwise, set $w^{jk} = -a_\delta$. Let $\phi^j = \sum_{k \neq j} w^{jk}$ which measures the difference between the effective neighbors of j accessed through the link j maintains and those accessed through incoming links. We claim that $\sum_{j \text{ in the cycle}} \phi^j \leqslant 0$. If this holds then there must be some player i such that $\phi^i \leqslant 0$, and we are done. To prove the claim we decompose ϕ^j as the sum of two parts: $\phi^{j,\text{cycle}} = \sum_{k \text{ in the cycle}} w^{jk}$ and $\phi^{j,\text{out}} = \sum_{k \text{ outside the cycle}} w^{jk}$. Clearly, $\phi^{j,\text{cycle}} \leqslant 0$; so we just need to show that $\sum_{j \text{ in the cycle}} \phi^{j,\text{out}} \leqslant 0$. To see why the latter holds, consider a player k outside the cycle and let l be the player in the cycle who is the root of the sub-tree that contains k. Note that $w^{l,k} \leqslant 0$. Let j_1 and j_2 be two players in the cycle each at distance $\delta \leqslant \overline{d}^{\text{cycle}}$ from l (also in the cycle), where $\overline{d}^{\text{cycle}}$ is the maximum distance between two players in the cycle. That is, j_1 and j_2 are equidistant and "opposite sides" from l. By construction $w^{j_1k} + w^{j_2k} = 0$. It follows that $\sum_{j \text{ in the cycle}} w^{jk} \leqslant 0$, and summing over all players k outside the cycle gives $\sum_{j \text{ in the cycle}} \phi^{j,\text{out}} \leqslant 0$. $\square$

Proof of Proposition 1. Let G^m denote the set of connected networks with m players and let $v(m) = \arg\max_{g \in G^m} \frac{1}{m} \sum_{i=1}^{m} \pi^i(g)$, the highest payoff per capita across networks in G^m. Suppose that an efficient network g^* has Q components and $m_1, \ldots, m_Q$ players in each of these components. Any component of size m in g^* must have a per capita welfare equal to $v(m)$. Hence, $W = \sum_{q=1}^{Q} m_q v(m_q)$ is social welfare associated to g^*. Note that $W \leqslant N \cdot \max_{m \in \{1,\ldots,N\}} v(m)$.

Write

$$\underline{v}(m) = \frac{m-1}{m}\left[f(m-2+a_1) - c\right] + \frac{1}{m} f(a_1(m-1))$$

for the average payoff of m agents organized in a star architecture. By hypothesis $N > f^{-1}(c + f(0))$, which implies $\underline{v}(N) > v(1)$. Thus the empty network is not efficient.

(i) If $a_1 = a_2$ and $a_\delta = 0$, $\delta \geqslant 3$ then the star is an efficient architecture in the set G^m. This is because any component must have at least $m - 1$ links, and a star attains the maximal benefits as all agents are connected to each other. Hence, $v(m) = \underline{v}(m)$ and the result is established by showing that $\underline{v}(m)$ is increasing. Indeed, using $f(a_1(m-1)) - f(m-2+a_1) < (a_1 - 1)(m - 2) f'(m-2+a_1)$ which follows by strict concavity, it is easy to show that $f'(m) \geqslant 2ca_1^2/m^2$ implies $\underline{v}'(m) > 0$ for all $m \geqslant 2$.

(ii) For each $m \geqslant 2$, let $\underline{z}(m) = \frac{m-1}{m}$, $\overline{z}(m) = \frac{m-1}{2}$ and define

$$\overline{v}(m) = \max_{z \in [\underline{z}(m), \overline{z}(m)]} f(m - 1 + 2z(a_1 - 1)) - cz. \tag{5}$$

The proof is structured in four steps.

Step 1: $\overline{v}(m) > v(m)$. Let x be the number of links in an efficient component of size $m \geqslant 2$. Let x^i respectively z^i denote the number of direct and indirect neighbors of player i in this component. Player i's benefits are bounded above by $f(z^i + a_1 x^i) \leqslant f(m - 1 + (a_1 - 1)x^i)$, which implies that total welfare in this component $mv(m) \leqslant \sum_i f(m - 1 + (a_1 - 1)x^i) - cx$. By the strict concavity of $f(.)$, using $\sum_i x_i = 2x$, and $m - 1 \leqslant x \leqslant m(m-1)/2$ we have

$$v(m) < f\left(m - 1 + (a_1 - 1)\frac{\sum_i x^i}{m}\right) - c\frac{x}{m} = f\left(m - 1 + 2(a_1 - 1)\frac{x}{m}\right) - c\frac{x}{m} \leqslant \overline{v}(m).$$

Step 2: $\overline{v}'(m) > \left(\frac{2ca_1^2}{(a_1 - 1/2)^2} - c\right)/m^2$ and in particular $\overline{v}(m)$ is strictly increasing.

The objective function of (5) is concave and the domain $[\underline{z}(m), \overline{z}(m)]$ shifts out as m increases. This implies that $\overline{v}(m)$ has three regions: (R1) for m small, $\overline{v}(m)$ is obtained as a corner solution at $z = \overline{z}(m)$; (R2) for intermediate m, the maximum is obtained as an interior solution; (R3) for m large the maximum is attained at $z = \underline{z}(m)$. For each of these cases, $\overline{v}'(m) > \left(\frac{2ca_1^2}{(a_1-1/2)^2} - c\right)/m^2$ can be verified directly, using $f'(n) \geqslant 2ca_1^2/n^2$, and the envelope theorem in the interior case.

Step 3: If $m > \max\{2M + 1 - a_1, 3a_1 - 2\}$ then $\overline{v}(m) = f(m - 1 + 2\underline{z}(m) \cdot (a_1 - 1)) - c\underline{z}(m)$. To see why, using $\underline{z}(m) \geqslant 1/2$ we have

$$f'(m - 1 + 2\underline{z}(m)(a_1 - 1)) \leqslant f'(m - 2 + a_1) < \frac{f(m-2+a_1) - f((m-2+a_1)/2)}{(m-2+a_1)/2} < \frac{2c}{m-2+a_1} \leqslant c,$$

when $m - 2 + a_1 > M$ and $m \geqslant 3a_1 - 2$ because of strong decreasing returns. But this chain of inequalities implies that (5) is maximized at $z = \underline{z}(m)$.

Step 4: From steps 1 and 2, the per capita payoff in the efficient network is at most $\overline{v}(N)$. Thereby, using step 3, the inefficiency of the star is bounded by

$$\begin{aligned}\overline{v}(N) - \underline{v}(N) &= f(N - 1 + 2\underline{z}(N)(a_1 - 1)) - f(N - 2 + a_1) \\ &\leqslant f(N - 3 + 2a_1) - f(N - 2 + a_1).\end{aligned}$$

By concavity, $f(N-3+2a_1) - f(N-2+a_1) \leqslant (a_1-1)f'(N-2+a_1)$ and Assumption 1 implies that $f'(N-2+a_1) < \frac{2c}{Na_1}$. Combining this with the previous yields $\overline{v}(N) - \underline{v}(N) < \frac{2c(a_1-1)/a_1}{N}$, as desired.

To establish the second statement, for each component $q \in \{1, \ldots, Q\}$ we have

$$\underline{v}(N) - v(m_q) \geqslant \overline{v}(N) - \overline{v}(m_q) - \frac{2c(a_1-1)/a_1}{N} \geqslant \left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right]\frac{(N-m_q)}{Nm_q} - \frac{2c(a_1-1)/a_1}{N},$$

where the last inequality follows from $\overline{v}(N) - \overline{v}(m_q) = \int_{m_q}^{N} \overline{v}'(n)dn \geqslant \left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right]\frac{(N-m_q)}{Nm_q}$. Multiplying by m_q and summing over q yields

$$N\underline{v}(N) - W \geqslant \left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right](Q - 1) - 2c(a_1 - 1)/a_1.$$

Since $N\underline{v}(N) \leqslant W$, then $\left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right](Q - 1) - \frac{2c(a_1-1)}{a_1} \leqslant 0$. Hence $Q \leqslant 1 + \frac{2(1-1/a_1)}{2a_1^2/(a_1-1/2)^2 - 1}$. □

Proof of Proposition 2. We introduce some terminology and notation. We say that i maintains a link to j if i pays at least $c/2$ for that link. For a network g and a link $(i, j) \in g$, let $S(g, ij) = \pi_i(g) + \pi_j(g) - \pi_i(g') + \pi_j(g')$ the surplus associated to link ij when the network is g, where $g' = g - (i, j)$ is the network that results from deleting the (i, j) link from g. For each $(x, y) \in \mathbb{R}_+^2$ we define $\sigma(x, y) = f(x) - f(0) + f(y + a_1) - f(y) - c$. Clearly, σ is increasing and concave in x and, by (MS), it is also increasing in y.

We begin by showing that the equilibrium network g^* is a tree. We first claim that no player maintains more than one link. Otherwise, for some link that i maintains, she gets at most half of her total effective neighbors through that link, while she is paying $c/2$ or more. Then dropping this link (setting t_j^i sufficiently negative) has a benefit of at least $c/2$ and a loss of at most $f(n) - f(n/2)$ where n is the number of i's effective neighbors. Hence under $(1, c/2)$ strong decreasing returns, it is optimal to drop the link. By the argument in Section 3, each component of the equilibrium network will have the directed tree and cycle architecture. Moreover, with $(1, c/2)$

strong decreasing returns, an equilibrium network never contains a cycle by the argument used in the proof of Lemma 5.

We next show that all terminal nodes maintain a link to the same player. Suppose that a terminal node t links i, i has access to m effective neighbors, and provides t access to q effective neighbors, where q includes the benefit i gets from her direct connection to i. Because $a_{\delta+1}/a_\delta$ is weakly decreasing, the inequality $q - a_1 \leqslant m/a_1$ must hold. This is because the $q - a_1$ effective neighbors that t has access to through i must all be in $N_d(i)$; but then their total value cannot exceed the total that accumulates to i herself (m) discounted by the highest relative discount rate $a_2/a_1 = 1/a_1$. It is also easy to verify that $S(g^*, t_1 i_1) = \sigma(q_1, m_1)$.

Towards a contradiction, suppose that there are two terminal nodes, t_1 and t_2, who link different players i_1 and i_2. Let $\overline{\delta} = d(t_1, i_2) = d(t_2, i_1)$, where $\overline{\delta} \geqslant 2$. Without loss of generality suppose that $S(g^*, t_1 i_1) \leqslant S(g^*, t_2 i_2)$. Consider the deviation where t_1 drops her link to i_1 and instead makes an offer to i_2, with a transfer amount chosen such that the payoff of t_1 after this move will increase slightly. If it is in t_2's best interest to accept this offer, then the previous allocation could not have been a pairwise equilibrium with unilateral status quo. We claim that this is a profitable pairwise deviation. To show this, we compute the surplus $S(\widetilde{g}, t_1 i_2)$ for the proposed link $t_1 i_2$, where $\widetilde{g} = g^* - (t_1, i_1) + (t_1, i_2)$. If $S(\widetilde{g}, t_1 i_2) > S(g^*, t_1 i_1)$ then the deviation is profitable: t_1 can enjoy a slightly higher payoff in the new profile, and at the same time, the payoff of i_2 increases relative to the status quo network $g^* - (t_1, i_1)$. To compute $S(\widetilde{g}, t_1 i_2)$ we account for the fact that, following the deviation, t_1 will no longer be linked to i_1, which might affect the payoff of i_2 who was potentially indirectly connected through i_1 to t_1 in network g^*. We have

$$\begin{aligned} S(\widetilde{g}, t_1 i_2) &\geqslant f(q_2 + a_2 - a_{\overline{\delta}+1}) - f(0) + f(m_2 + 2a_1 - a_{\overline{\delta}}) - f(m_2 + a_1 - a_{\overline{\delta}}) - c \\ &= \sigma(q_2 + a_2 - a_{\overline{\delta}+1}, m_2 + a_1 - a_{\overline{\delta}}). \end{aligned}$$

The first term here is a bound for the benefit accumulating to player t_1, accounting for the fact that t_1 is no longer connected to i_1, which reduces q_2 by $a_{\overline{\delta}+1}$. The second and third terms measure the change in the benefits to i_2, who is now directly connected to t_1, but lost the indirect connection, which costs her $a_{\overline{\delta}}$ effective neighbors. Since $S(g^*, t_1 i_1) \leqslant S(g^*, t_2 i_2)$, the claim is established if we show that $S(\widetilde{g}, t_1 i_2) > S(g^*, t_2 i_2)$ or $\sigma(q_2 + a_2 - a_{\overline{\delta}+1}, m_2 + a_1 - a_{\overline{\delta}}) > \sigma(q_2, m_2)$. Since σ is concave in its first argument, this inequality becomes harder to satisfy if we increase q_2. From above, $q_2 \leqslant a_1 + m_2/a_1$. Hence, if we let $n = 1 + m_2/a_1$, substitute in $q_2 = n - 1 + a_1$ and use $a_2 = 1$ we get the condition that $\sigma(n + a_1 - a_{\overline{\delta}+1}, na_1 - a_{\overline{\delta}}) > \sigma(n - 1 + a_1, (n-1)a_1)$. To see why this inequality holds, note that $a_1 a_{\overline{\delta}+1} \leqslant a_{\overline{\delta}}$ by assumption and hence $f((n+1)a_1 - a_{\overline{\delta}}) - f(na_1 - a_{\overline{\delta}}) \geqslant f((n+1)a_1 - a_1 a_{\overline{\delta}+1}) - f(na_1 - a_1 a_{\overline{\delta}+1})$. Then we can write

$$\begin{aligned} \sigma(n + a_1 - a_{\overline{\delta}+1}, na_1 - a_{\overline{\delta}}) &\geqslant \sigma(n + a_1 - a_{\overline{\delta}+1}, na_1 - a_1 a_{\overline{\delta}+1}) \\ &> \sigma(n - 1 + a_1, (n-1)a_1), \end{aligned}$$

where the last step uses the fact that σ is increasing in its second argument. This gives the desired inequality, verifying that all terminal nodes connect to the same player.

It follows that g^* consists of a star and a path starting from the center of this star, player i. We show that the length of this path is zero. First note that, including i, the path has at most $2d$ players. Otherwise, if the path has $2d + 1$ or more players there would be a player j in the middle of the path such that $N_d(j)$ only includes a line of d players going "forward" as well as a line of d players going "backward" along the path. By $(1, c)$-strong decreasing returns, j would optimally drop her link. We conclude that that player i has at least $N - 2d$ incoming links from players in the star. Thus, since $(2d-1)a_1 < (N-2d)a_1$ or equivalently $N > 4d-1$ holds by assumption, player i has no incentive to form a link. Therefore the path has length zero.

Finally, consider the payoffs in the star. The benefit of the center from each link is $x = f((N-1)a_1) - f((N-2)a_1)$, and hence the center pays a transfer of at most this amount for any link. His total link cost is thus at most $(N-1)x$. The link cost of an agent on the periphery is at least $c-x$. The total benefit advantage of the center relative to an agent in the periphery is $f((N-1)a_1) - f(a_1 + (N-2)a_2)$; this has to be greater than $Nx - c$ for the center to earn the highest payoff. Combining strong decreasing returns and concavity, we obtain $x(N-1)/2 \leqslant f((N-1)a_1) - f((N-1)a_1/2) < c/2$, which implies $(N-1)x < c$, or $Nx - c < x$. Hence the center will earn the highest payoff when $f((N-1)a_1) - f(a_1 + (N-2)a_2) \geqslant x$, or equivalently when $f(a_1 + (N-2)a_2) \leqslant f((N-2)a_1)$, which holds as long as $N \geqslant 3 + 1/(a_1 - 1)$. $\square$

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